JOURNAL OF ALGEBRA 130, 235–259 (1990)

On Quotients of Hom-Functors and Representations of Finite General Linear Groups, I

RICHARD DIPPER*

Department of Mathematics, The University of Oklahoma, Norman, Oklahoma 73019

Communicated by Michel Broué

Received June 15, 1988

1. INTRODUCTION

This is the first in a series of papers on quotients of Hom-functors and representations of general linear groups. The overall aim of this series is threefold: First to put the theory developed in [5, 6, 12–14, 7, 8] on common ground and to present much simpler and more elegant new proofs of the main results in these papers. Second, we shall extract general background material which will make the theory more accessible to generalization to other families of groups of Lie type. The third objective is to unify the representation theory of symmetric groups and of general linear groups in the describing and non-describing characteristic case.

In this first paper we shall develop terminology and background material on quotients of Hom-functors, that is, functors of the form $\mathrm{Hom}(P, -)/\mathrm{Hom}(P, -)J$, where P is projective and J is a certain ideal of the endomorphism ring of P. The Hom-functors $\mathrm{Hom}(P, -)$ certainly play an important role in representation theory. Some of their properties have been extensively investigated by Auslander in [2, 3], and we shall generalize some of the results there.

Schur functors from the category of homogeneous polynomial representations of general linear groups of fixed degree r (or equivalently the module category of a certain Schur algebra) to the module category of the symmetric groups S_r are special functors of the form $\mathrm{Hom}(P, -)$ (see, e.g., [9]).

The reader might care to bear in mind the following example which illuminates why the general results on quotients of Hom-functors in this paper are important for the representation theory of general linear groups

* The author gratefully acknowledges support received from the Deutsche Forschungsgemeinschaft and from NATO under Grant 0222/87.

0021-8693/90 $3.00

in the non-describing characteristic case and how they connect to the describing characteristic case.

Let $G = GL_n(q)$ for some prime power q and $n \in \mathbb{N}$. Let $(F, \mathcal{O}, K)$ be a split p-modular system for G, where p does not divide q. So $\mathcal{O}$ is a rank one discrete complete valuation ring with quotient field K of characteristic 0 and residue field F of characteristic p, and both F and K are splitting fields for all subgroups of G. We denote the (Borel) subgroup of all upper triangular matrices in G by B and the trivial RB-module ($R = F$, $\mathcal{O}$, or K) by R_B. The endomorphism ring $\mathscr{H}_R$ of the induced module $R_B^G = R_B \otimes_{RB} RG$ is the Hecke algebra over R associated with the Weyl group $W \cong S_n$ of G. In particular it is free as R-module with basis $\{T_w \mid w \in W\}$, and the multiplication of these basis elements involves polynomials in q with integer coefficients. (For $R = K$, these relations have been determined by Iwahori [11]; for the general ase see [5].) Let $B = TU$ be the Levi decomposition of B, so T is the torus consisting of all diagonal matrices in G, and U is the set of unipotent upper triangular matrices. Since p does not divide q, U is p-regular, and the projective cover $P_{\mathcal{O}}$ of $\mathcal{O}_B$ is the projective cover of the trivial T-module lifted to B along the canonical epimorphism from B onto T with kernel U. As a consequence $P_K = K \otimes_{\mathcal{O}} P_{\mathcal{O}}$ decomposes into one dimensional KB-modules, each occurring with multiplicity one. The following result can be deduced easily from Frobenius reciprocity, but also follows directly from character theory of general linear groups [10] (or in the language of Deligne Lusztig theory from the orthogonality relations for Deligne Lusztig operators; see, e.g., [16, 6.14]).

1.1. LEMMA. *Let χ be a non-trivial one dimensional constituent of P_K. Then χ^G and K_B^G have no irreducible constituent in common.*

Let β: $P_{\mathcal{O}}^G \to \mathcal{O}_B^G$ be the map induced from an epimorphism from $P_{\mathcal{O}}$ onto $\mathcal{O}_B$. So we have an exact sequence

$$0 \to \ker \beta \to P_{\mathcal{O}}^G \xrightarrow{\beta} \mathcal{O}_B^G \to 0.$$

We have the following:

1.2. COROLLARY. *$K \otimes_{\mathcal{O}} \ker \beta$ and $K_B^G = K \otimes \mathcal{O}_B^G$ have no irreducible constituent in common.*

That is precisely the situation that we are dealing with in this paper. We shall show in (4.7) and (2.1) that β induces a canonical epimorphism from $\mathscr{E}_R = \mathrm{End}_{RG}(P_R)$, onto $\mathscr{H}_R = \mathrm{End}_{RG}(R_B^G)$, where $P_R = R \otimes_{\mathcal{O}} P_{\mathcal{O}}$. The kernel J_{β_R} of this epimorphism consists of all endomorphisms of P whose image is contained in $\ker \beta_R$, that is, the right annihilator of β_R in $\mathscr{E}_R$, where $\beta_R = 1_R \otimes_{\mathcal{O}} \beta$. As a consequence of (4.10) we have that the decomposition matrix of $\mathscr{H}_{\mathcal{O}}$ is a submatrix of the decomposition matrix of G. The functor $H_R = \operatorname{Hom}_{RG}(P_R, -)/\operatorname{Hom}_{RG}(P_R, -)J_{\beta_R}$ from the category of RG-modules into the category of $\mathscr{H}_R$-modules is the link between the representations theory of general linear groups (in non-describing characteristic) and Hecke algebras associated with symmetric groups. However, there are more consequences of (1.2) which are described briefly below:

If λ is a partition of n let W_λ be the corresponding standard parabolic (Young) subgroup of W and take $x_\lambda \in \mathscr{H}_R$ to be $x_\lambda = \sum_{w \in W_\lambda} T_w$. Let $M_{\mathcal{O}}$ be the $\mathscr{H}_{\mathcal{O}}$-module

$$\bigoplus_{\lambda \vdash n} x_\lambda \mathscr{H}_{\mathcal{O}},$$

where $\lambda \vdash n$ means λ is a partiton of n.

Corollary (2.23) gives an isomorphism between the q-Schur algebra $\mathscr{S}_R = \operatorname{End}_{\mathscr{H}_R}(M_R)$ and the endomorphism ring of the RG-module

$$\bigoplus_{\lambda \vdash n} x_\lambda R_B^G \qquad \text{(compare [8, 2.24])}.$$

Replacing $x_\lambda \in \mathscr{H}_R$ by $y_\lambda = \sum_{w \in W_\lambda} (-q)^{-l(w)} T_w$ ($l(x)$ denoting the length of $w \in W$) and applying (4.17) will produce another RG-module with endomorphism ring $\mathscr{S}_R$, which can be shown to satisfy again (4.7). So we may apply the theory developed in this paper.

This will be done in the second of this series of papers. More generally we shall replace $B \leqslant G$ by parabolic subgroups P of G, and the trivial B-module by cuspidal P-modules. Corollary (4.10) then will relate the decomposition numbers of $\mathcal{O}G$, $\mathscr{H}_{\mathcal{O}}$, and $\mathscr{S}_{\mathcal{O}}$ reproving in particular [8, 4.9], and we shall recover the main results of the papers cited above.

2. Quotients of Hom-Functors

We begin by collecting some basic properties of quotients of Hom-functors $\operatorname{Hom}_T(P, -)$ for a ring T and a projective T-module P. We shall impose some restrictions on T and P to avoid lengthy and complicated hypotheses. However, as the reader might check easily, many of the results in this first section hold in more general circumstances. Henceforth assume that T is an R-algebra, finitely generated as R-module, where R is a Noetherian commutative ring. Moreover we assume that T has a multiplicative identity as well as R has, and that T is unital as R-module. In particular we may identify the identities of R and T. All occurring modules are, if not stated otherwise, unital finitely generated right modules, and we denote by mod T the category of all unital, finitely generated right

T-modules. To ensure that every M in mod T has a projective cover we assume that T is semiperfect. The applications we have in mind involve group rings over fields or discrete complete valuation rings, or related algebras, which always will satisfy the assumptions above.

For a T-module M a *projective presentation* of M is a projective T-module P together with an epimorphism $P \xrightarrow{\beta} M$. We then define $\mathscr{E}_\beta$ to be the ring of all endomorphisms φ of P leaving the kernel $\ker\beta$ of β invariant. Note that all modules involved are finitely generated as R-modules, in particular the R-algebras $\mathscr{H} = \mathrm{End}_T(M)$, $\mathscr{E} = \mathrm{End}_T(P)$, and $\mathscr{E}_\beta$. The canonical algebra homomorphism $\tilde{\beta}\colon \mathscr{E}_\beta \to \mathscr{H}$ induced by β is surjective, since P is projective, and has kernel $J_\beta = \ker\tilde{\beta} = \{\varphi \in \mathscr{E} \mid \varphi(P) \subseteq \ker\beta\}$. So we have:

2.1. *J_β is an ideal of $\mathscr{E}_\beta$ and $\mathscr{E}_\beta/J_\beta \cong \mathscr{H}$ as R-algebra canonically.*

2.2. DEFINITION. Let $M \in \mathrm{mod}\, T$, and let $\mathscr{H} = \mathrm{End}_T(M)$ acting on M on the left. We define the functor

$$H = H_M\colon \mathrm{mod}\, T \to \mathrm{mod}\, \mathscr{H}$$

as follows: First we choose a projective presentation $P \xrightarrow{\beta} M$ of M. For $V \in \mathrm{mod}\, T$ we define $H(V)$ to be the right $\mathscr{H}$-module

$$H(V) = \mathrm{Hom}_T(P, V)/\mathrm{Hom}_T(P, V)J_\beta,$$

where $\mathrm{Hom}_T(P, V)$ is considered as right $\mathscr{E}_\beta$-module via the left $\mathscr{E}_\beta$-module structure of P derived from restricting the left $\mathscr{E}$-module structure on P to the subalgebra $\mathscr{E}_\beta$ of $\mathscr{E}$. We remark that $H(V)$ is finitely generated, so $H(V) \in \mathrm{mod}\, \mathscr{H}$; this follows easily from the fact that T is finitely generated as R-module and M, P are finitely generated as T-modules (see, e.g., [4, Ex. 2.6]).

If $V' \in \mathrm{mod}\, T$ and if $f\colon V \to V'$ is T-linear, then $\mathrm{Hom}_T(P, f)$: $\mathrm{Hom}_T(P, V) \to \mathrm{Hom}_T(P, V')$ mapping $\varphi \in \mathrm{Hom}_T(P, V)$ to $f\varphi \in \mathrm{Hom}_T(P, V')$ is $\mathscr{E}$- hence $\mathscr{E}_\beta$-linear. So it induces an $\mathscr{H}$-linear map $H(f)$ from $H(V)$ into $H(V')$. Obviously $H = H_M$ is an additive covariant functor preserving direct sums. Moreover it can be seen easily that H is right exact.

2.3. *Remark.* For $X \in \mathrm{mod}\, T$ let the dual module $X^\#$ be the left T-module $\mathrm{Hom}_T(X, T)$, where the left action of T on $X^\#$ is induced by considering T as left T-module. P is finitely generated, therefore $\mathrm{Hom}_T(P, V) \cong V \otimes_T P^\#$ as right $\mathscr{E}$-modules (see, e.g., [4, 2.32]). So defining the T–$\mathscr{H}$-bimodule ${}_TX_\mathscr{H}$ to be $P^\# \otimes_{\mathscr{E}_\beta} \mathscr{H}$ (considering $\mathscr{H}$ as left $\mathscr{E}_\beta$-module, via (2.1)), the functor H_M is equivalent to the functor $- \otimes_T X_\mathscr{H}$. In particular we may consider H_M as the composite of three functors: First

$\mathrm{Hom}_T(P, -)$: $\mathrm{mod}\, T \to \mathrm{mod}\, \mathscr{E}$, then the restriction functor from $\mathrm{mod}\, \mathscr{E}$ to $\mathrm{mod}\, \mathscr{E}_\beta$, finally the functor $- \otimes_{\mathscr{E}_\beta} \mathscr{H}$, factoring out $J_\beta \trianglelefteq \mathscr{E}_\beta$.

The following lemma can be shown using routine arguments:

2.4. LEMMA. *Let* $M \in \mathrm{mod}\, T$. *Then* H_M *defined in* (2.2) *depends only on* M, *not on the projective presentation of* M *chosen in* (2.2) (*up to isomorphisms of functors*).

2.5. EXAMPLE. If $M = P \in \mathrm{mod}\, T$ is projective (and β is the identity mapping), then $\mathscr{E}_\beta = \mathrm{End}_T(P)$ and $H_M = \mathrm{Hom}_T(P, -)$. This case was studied in detail by M. Auslander in [2, 3] and we shall use his work as a guideline in our investigation of the functor H_M generalizing some of the results in [2].

All the upcoming applications satisfy $\mathscr{E}_\beta = \mathscr{E}$ in the notation of (2.2), so our special interest is with this case and we will henceforth assume $\mathscr{E}_\beta = \mathscr{E}$. We collect all our assumptions in the following:

2.6. *Hypothesis.* R is a commutative Noetherian ring, and T a semiperfect R-algebra finitely generated as R-module. $M \in \mathrm{mod}\, T$, $\mathscr{H} = \mathrm{End}_T(M)$, and $P \xrightarrow{\beta} M$ is a projective presentation of M such that $\mathscr{E}_\beta = \mathscr{E} = \mathrm{End}_T(P)$, that is, $\ker \beta$ is invariant under the action of $\mathscr{E}$. The ideal $\{\varphi \in \mathscr{E} \mid \varphi(P) \subseteq \ker \beta\}$ is denoted by J_β, and $\mathscr{H}$ is idendified with $\mathscr{E}|J_\beta$ by the canonical isomorphism induced by β. The functor $H = H_M$: $\mathrm{mod}\, T \to \mathrm{mod}\, \mathscr{H}$ is defined by (2.2).

Obviously we can find all functors satisfying (2.6) by taking $M = P/U$, where $P \in \mathrm{mod}\, T$ is projective and U is an $\mathscr{E}$–T-subbimodule of P.

The condition that $\ker \beta$ is invariant under the action of $\mathscr{E}$ is not independent of the choice of the projective presentation $P \xrightarrow{\beta} M$. For example, if $M = P/U$, where U is an $\mathscr{E}$–T-subbimodule of the projective T-module P, and if β: $P \to M$ is the canonical epimorphism, we may choose $Q = P \oplus P$ and $\tilde{\beta}$: $Q \to M$ to be $\tilde{\beta} = \beta \oplus 0$. Obviously, $\ker \tilde{\beta}$ is not invariant under the action of $\mathrm{End}_T(Q)$. However, the following lemma is easy to prove:

2.7. LEMMA. *Suppose* (2.6) *holds for* $P \xrightarrow{\beta} M$. *Let* $Q \xrightarrow{\alpha} M$ *be a projective cover of* M. *Then* (2.6) *holds for* $Q \xrightarrow{\alpha} M$ *as well, that is,* $\ker \alpha$ *is invariant under the action of* $\mathrm{End}_T(Q)$.

Assume (2.6). Applying the exact functor $\mathrm{Hom}_T(P, -)$ to the short exact sequence

$$0 \to \ker \beta \to P \xrightarrow{\beta} M \to 0$$

we get the short exact sequence of $\mathscr{E}$–$\mathscr{E}$-bimodules (considering M as left $\mathscr{E}$-module via the epimorphism $\mathscr{E} \to \mathscr{H}$ induced by β)

$$0 \to J_\beta \to \mathscr{E} \to \mathrm{Hom}_T(P, M) \to 0$$

using that $\mathrm{Hom}_T(P, \ker\beta) = J_\beta$. As a consequence J_β annihilates $\mathrm{Hom}_T(P, M)$ from both sides. Obviously $H_M(P) = \mathrm{Hom}_T(P, P)/\mathrm{Hom}_T(P, P) J_\beta = \mathscr{E}/J_\beta = \mathscr{H}$ ad $\mathscr{H}$–$\mathscr{H}$-bimodules and we have shown:

2.8. Lemma. *Suppose* (2.6). *Then* $\ker\beta \leqslant \ker\varphi$ *for all* $\varphi \in \mathrm{Hom}_T(P, M)$ *and* $H_M(M) = \mathrm{Hom}_T(P, M) \cong \mathscr{H} \cong \mathscr{E}/J_\beta = H_M(P)$ *as* $\mathscr{H}$–$\mathscr{H}$ *bimodules.*

2.9. Definition [2, Sect. 5]. Let $P, V \in \mathrm{mod}\, T$ and let P be projective. Then the *P-torsion submodule* $t_P(V)$ is the sum of all submodules X of V with $\mathrm{Hom}_T(P, X) = (0)$. The *kernel* $\mathbf{ker}P$ of P is the full subcategory of mod T, whose objects are the T-modules V with $\mathrm{Hom}_T(P, V) = (0)$. So $V \in \mathbf{ker}\, P$ if and only if $t_P(V) = V$.

2.10. Lemma. *Let* $P \in \mathrm{mod}\, T$ *be projective,* $V \in \mathrm{mod}\, T$. *Then* $t_P(V)$ *is the unique maximal submodule* X *of* V *such that* $\mathrm{Hom}_T(P, X) = (0)$. *Moreover,* $t_P(V/t_P(V)) = (0)$ *and for a homomorphism* $f\colon V \to V'$ $(V, V' \in \mathrm{mod}\, T)$, *we have* $f(t_P(V)) \subseteq t_P(V')$.

Proof. This follows from [2, Sect. 5], but can be also seen directly as follows: Since all modules involved are finitely generated (by general assumption) it is enough to show that $\mathrm{Hom}_T(P, X_1 + X_2) = (0)$, provided $\mathrm{Hom}_T(P, X_i) = (0)$ for $i = 1, 2$ $(X_1, X_2 \leqslant V)$. By projectivity of P, $\mathrm{Hom}_T(P, X_2) = (0)$ implies $\mathrm{Hom}_T(P, X_2/(X_1 \cap X_2)) = (0)$, hence $\mathrm{Hom}_T(P, (X_1 + X_2/X_1)) = (0)$, so $\mathrm{im} f \subseteq X_1$ for any T-linear map $f\colon P \to X_1 + X_2$. Since $\mathrm{Hom}_T(P, X_1) = (0)$, f is the zero map and we have shown $\mathrm{Hom}_T(P, X_1 + X_2) = (0)$. So $t_P(V) \in \mathbf{ker}\, P$. Now P being projective implies easily that $t_P(V/t_P(V)) = (0)$, and that $f(t_P(V)) \subseteq t_P(V')$ for any T-linear mapping $f\colon V \to V'(V, V' \in \mathrm{mod}\, T)$.

2.11. Definition. Define the functor $A_P\colon \mathrm{mod}\, T \to \mathrm{mod}\, T$ by $A_P(V) = V/t_P(V)$ and $A_P(f)$ to be the induced homomorphism from $V/t_P(V)$ to $V'/t_P(V')$ for any T-linear map $f\colon V \to V'$ in mod T. We say V is *P-torsionless* if $t_P(V) = (0)$. So in particular for $V \in \mathrm{mod}\, T$, $A_P(V)$ is P-torsionless.

Obviously for $V \in \mathrm{mod}\, T$ we have $\mathrm{Hom}_T(P, V) \cong \mathrm{Hom}_T(P, A_P(V))$ as $\mathscr{E}$-modules ($\mathscr{E} = \mathrm{End}_T(P)$), and we have shown:

2.12. Lemma. *Assume* (2.6) with $P \xrightarrow{\beta} M$. *Then* $H_M(V) = H_M(A_P(V))$

for all $V \in \operatorname{mod} T$. *Similarly if* $f\colon V \to V'$ *is* T-*linear* ($V, V' \in \operatorname{mod} T$), *then* $H_M(f) = H_M(A_P(f))$. *So* $H_M = H_M \circ A_P$.

2.13. DEFINITION. Assume (2.6), we define four functors from mod $\mathscr{H}$ to mod T as follows: (i) $F_M = - \otimes_{\mathscr{H}} M$, (ii) $\tilde{F}_M = A_P \circ F_M$, (iii) $G_M = - \otimes_{\mathscr{E}} P$, (iv) $\hat{G}_M = A_P \circ G_M$.

2.14. *Remarks.* Considering M as left $\mathscr{E}$-module, $- \otimes_{\mathscr{E}} M$ is a functor from mod $\mathscr{E}$ into mod T. By general theory this functor factors through the canonical functor mod $\mathscr{E} \to$ mod $\mathscr{H}$ given by $X \mapsto X/XJ_\beta$. In particular F_M equals the restriction of $- \otimes_{\mathscr{E}} M$ to the subcategory mod $\mathscr{H}$, and general theory implies that indeed $G_M = - \otimes_{\mathscr{H}} (P/J_\beta P)$. Obviously $J_\beta P \leqslant \ker \beta$ as $\mathscr{E}$–T-bimodule (recall that $J_\beta = \operatorname{Hom}_T(P, \ker \beta)$), so, for $X \in \operatorname{mod} \mathscr{H}$, we have an epimorphism from $G_M(X) = X \otimes_{\mathscr{H}} (P/J_\beta P)$ onto $F_M(X) = X \otimes_{\mathscr{H}} M$ whose kernel is the image of $X \otimes_{\mathscr{H}} (\ker \beta / J_\beta P)$ in $G_M(X) = X \otimes_{\mathscr{H}} (P/J_\beta P)$. We remark that $J_\beta P$ is the submodule of $\ker \beta$ generated by all images of homomorphisms from P to $\ker \beta$, such that in particular $\ker \beta/(J_\beta P) \in \mathbf{ker}\, P$, and $J_\beta P$ is the trace $\tau_P(\ker \beta)$ of P in $\ker \beta$ according to the following definition:

2.15. DEFINITION (Compare [2, Sect. 6]). Let $P \in \operatorname{mod} T$ be projective, $\mathscr{E} = \operatorname{End}_T(P)$, $V \in \operatorname{mod} T$, and let $X \subseteq \operatorname{Hom}_T(P, V)$. Then the *P-trace* XP of X in V is the T-submodule of V generated by the images of all homomorphisms $\varphi\colon P \to V$ belonging to X. If $X = \operatorname{Hom}_T(P, V)$, we denote XP by $\tau_P(V)$ and call it the *trace* of P in V.

2.16. LEMMA. *Assume* (2.6) . *Let* $\check{H}_M$ *be one of the functors defined in* (2.13). *Then* $\check{H}_M$ *is a right inverse of* H_M, *that is, for* $X \in \operatorname{mod} \mathscr{H}$, *we have* $H_M(\check{H}_M(X)) \cong X$ *as* $\mathscr{H}$-*module, and this isomorphism is natural in* X.

Proof. By general theory (see, for example, [1, 20.10]), since P is finitely generated projective, and by (2.14), (2.8), $\operatorname{Hom}_T(P, X \otimes_{\mathscr{H}} M) \cong \operatorname{Hom}_T(P, X \otimes_{\mathscr{E}} M) \cong X \otimes_{\mathscr{E}} \operatorname{Hom}_T(P, M) \cong X \otimes_{\mathscr{E}} \mathscr{H} \cong X \otimes_{\mathscr{H}} \mathscr{H} \cong X$ as $\mathscr{H}$-module. In particular $\operatorname{Hom}_T(P, X \otimes_{\mathscr{H}} M) J_\beta = (0)$, hence $H_M(F_M(X)) \cong X$ as desired. All isomorphisms involved are natural, so the lemma follows for $\check{H}_M = F_M$. By (2.12), $H_M = H_M \circ A_P$, so it holds for $\check{H}_M = \tilde{F}_M$ as well. The cases $\check{H}_M = G_M$ respectively $\check{H}_M = \tilde{G}_M$ are proved similarly.

2.17. LEMMA. *Assume* (2.6), *and let* $V \in \operatorname{mod} T$.

(i) $H_M(V) \cong H_M(\tau_P(V))$ *canonically.*

(ii) *If* X *is a right* $\mathscr{E}$-*submodule of* $\operatorname{Hom}_T(P, V)$, *then* $\operatorname{Hom}_T(P, XP) \cong X$ *as* $\mathscr{E}$-*modules.*

Proof. Part (i) follows immediately from [2, 6.1] and (2.2). To prove (ii) we consider the canonical mapping $X \otimes_{\mathscr{E}} P \xrightarrow{\gamma} V$ induced by $\gamma(x \otimes u) = x(u)$ for $x \in X$, $u \in P$. Note that γ is a T-linear epimorphism from $X \otimes_{\mathscr{E}} P$ onto $XP \leqslant V$. So, by projectivity of P, the $\mathscr{E}$-linear mapping $\gamma^* = \operatorname{Hom}_T(P, \gamma)$: $\operatorname{Hom}_T(P, X \otimes_{\mathscr{E}} P) \to \operatorname{Hom}_T(P, XP)$ is an epimorphism. We also have a canonical mapping $X \cong X \otimes_{\mathscr{E}} \operatorname{Hom}_T(P, P) \xrightarrow{\sigma} \operatorname{Hom}_T(P, X \otimes_{\mathscr{E}} P)$ given by $\sigma(x) = \varphi_x$, where $\varphi_x \in \operatorname{Hom}_T(P, X \otimes_{\mathscr{E}} P)$ maps $u \in P$ to $x \otimes u$. By standard arguments (see, e.g., [1, 20.10]) σ is an isomorphism of $\mathscr{E}$-modules, since P is finitely generated projective. Now $(\gamma^*(\varphi_x))(u) = \gamma(\varphi_x(u)) = \gamma(x \otimes u) = x(u)$ for all $u \in P$ and for any $x \in X$, so $\gamma^*\sigma(x) = x$ for all $x \in X$. So $X \subseteq \operatorname{Hom}_T(P, XP)$ is the image of the epimorphism $\gamma^*\sigma$, that is, $X = \operatorname{Hom}_T(P, XP)$ as desired.

We will apply the previous lemma to the following situation.

2.18. COROLLARY. *Suppose Y is an $\mathscr{H}$-submodule of $H_M(V) = \operatorname{Hom}_T(P, V)/\operatorname{Hom}_T(P, V)J_\beta$. Let $\tilde{Y}$ be the full preimage of Y in $\operatorname{Hom}_T(P, V)$. Then $H_M(\tilde{Y}P) = Y$.*

In the notation of (2.17) let Z be the kernel of $\gamma: X \otimes_{\mathscr{E}} P \to XP$, so we have an exact sequence

$$0 \longrightarrow Z \longrightarrow X \otimes_{\mathscr{E}} P \xrightarrow{\gamma} XP \to 0.$$

Applying the exact functor $\operatorname{Hom}_T(P, -)$ results in an exact sequence

$$0 \longrightarrow \operatorname{Hom}_T(P, Z) \longrightarrow \operatorname{Hom}_T(P, X \otimes_{\mathscr{E}} P) \xrightarrow{\gamma^*} \operatorname{Hom}_T(P, XP) \longrightarrow 0.$$

However, in the notation of (2.17), $\gamma^*\sigma$ and σ are isomorphisms, so γ^* is an isomorphism as well and conselquently $\operatorname{Hom}_T(P, Z) = (0)$. So we have shown:

2.19. COROLLARY. *Assume* (2.6), *and let $V \in \operatorname{mod} T$. Let X be an $\mathscr{E}$-submodule of* $\operatorname{Hom}_T(P, V)$. *Let Z be the kernel of the canonical T-linear map $\gamma: X \otimes_{\mathscr{E}} P \to XP \leqslant V$. Then $Z \in$ **ker** P, so $Z \leqslant t_P(X \otimes_{\mathscr{E}} P)$, and γ induces an isomorphism between $A_P(X \otimes_{\mathscr{E}} P)$ and $A_P(XP)$. So, if V (and therefore XP as well) is P-torsionless, then $\tilde{G}_M(X) = XP$.*

Two special cases are of interest:

2.20. COROLLARY. *Assume* (2.6).

(i) *Let X be a right ideal of $\mathscr{E}$. Then* $\operatorname{Tor}_1^{\mathscr{E}}(X, P)$, *the kernel of the canonical mapping $X \otimes_{\mathscr{E}} P \to XP$, belongs to* **ker** *$P$, $\tilde{G}_M(X) = A_P(XP)$, and* $\operatorname{Hom}_T(P, XP) \cong X$.

(ii) *Let Y be a right ideal of $\mathscr{H}$. Then* $\operatorname{Tor}_1^{\mathscr{H}}(Y, M)$, *the kernel of the canonical mapping $Y \otimes_{\mathscr{H}} M \to YM$, belongs to* **ker** P, $\tilde{F}_M(Y) = A_P(YM)$, *and $H_M(YM) = Y$. If in particular M is P-torsionless, then $t_P(T \otimes_{\mathscr{H}} M) =$* $\operatorname{Tor}_1^{\mathscr{H}}(Y, M)$ *and* $\tilde{F}_M(Y) = YM$.

Proof. Part (i) is a special case of (2.17). To prove (ii) we proceed as in (2.17): First we note that the canonical isomorphism $Y \cong Y \otimes_{\mathscr{H}} \operatorname{Hom}_T(P, M)$ is explicitly given by $y \mapsto y \otimes \beta$ for $y \in Y$, where $\beta: P \to M$ is given in (2.6). Again by [1, 20.10], $\sigma: Y \otimes_{\mathscr{H}} \operatorname{Hom}_T(P, M) \to \operatorname{Hom}_T(P, Y \otimes_{\mathscr{H}} M)$ is an isomorphism, and is given by $\sigma(y \otimes f) = \varphi_{y,f}$ with $\varphi_{y,f}(u) = y \otimes f(u) \in Y \otimes_{\mathscr{H}} M$ for $y \in Y$, $f \in \operatorname{Hom}_T(P, M)$, and $u \in P$. Using the epimorphism $\gamma^*: \operatorname{Hom}_T(P, Y \otimes_{\mathscr{H}} M) \to \operatorname{Hom}_T(P, YM)$ induced by the canonical epimorphism $Y \otimes_{\mathscr{H}} M \xrightarrow{\gamma} YM$ and proceeding as in (2.17) we see that $\operatorname{Hom}_T(P, YM) \cong Y$ as $\mathscr{E}$-modules canonically. In particular $\operatorname{Hom}_T(P, YM) J_\beta = (0)$, hence $\operatorname{Hom}_T(P, YM) = H_M(Y) \cong Y$ as $\mathscr{H}$-modules. Tensoring the exact sequence $0 \to Y \to \mathscr{H} \to \mathscr{H}/Y \to 0$ with the left $\mathscr{H}$-module M yields $\ker \gamma = \operatorname{Tor}_1^{\mathscr{H}}(Y, M)$. Now (ii) follows.

2.21. THEOREM. *Assume* (2.6) *and let X, Y be right ideals of $\mathscr{H}$. Suppose M is P-torsionless. Then H_M induces an isomorphism, also denoted by H_M, from* $\operatorname{Hom}_T(XM, YM)$ *onto* $\operatorname{Hom}_{\mathscr{H}}(X, Y)$.

Proof. H_M induces a homomorphism $H_M: \operatorname{Hom}_T(XM, YM) \to \operatorname{Hom}_{\mathscr{H}}(X, Y)$ by functionality of H_M, identifying $H_M(XM)$ and X respectively $H_M(YM)$ and Y using (2.20). Obviously H_M is R-linear. Let $\varphi: X \to Y$ be $\mathscr{H}$-linear. By (2.20), $XM \cong A_P(X \otimes_{\mathscr{H}} M)$ and $YM \cong A_P(Y \otimes_{\mathscr{H}} M)$. Let $\psi: XM \to YM$ be the T-linear map which is induced by $\varphi \otimes \mathrm{id}_M$: $X \otimes_{\mathscr{H}} M \to Y \otimes_{\mathscr{H}} M$, where $\mathrm{id}_M: M \to M$ is the identity mapping. By (2.12) and (2.16), $H_M(\psi) = \varphi$. So H_M is surjective. Let $f: XM \to YM$ be T-linear, and let U be the image $\operatorname{im} f$ of f. Suppose $U \neq (0)$. Since M is P-torsionless, the same holds for U, so $\operatorname{Hom}_T(P, U) \neq (0)$. By (2.8), $\operatorname{Hom}_T(P, M) \cong \mathscr{H}$ hence $\operatorname{Hom}_T(P, M) J_\beta = (0)$. So, since $\operatorname{Hom}_T(P, U) \leqslant \operatorname{Hom}_T(P, M)$, we have $\operatorname{Hom}_T(P, U) J_\beta = (0)$ as well, hence $H_M(U) = \operatorname{Hom}_T(P, U)$. Let $0 \neq \rho \in \operatorname{Hom}_T(P, U)$. Then since $f: XM \to U$ is surjective and P is projective we find $\tau \in \operatorname{Hom}_T(P, XM) = H_M(XM)$ such that $(H_M(f))(\tau) = \operatorname{Hom}_T(P, f)(\tau) = f\tau = \rho \neq 0$. In particular $H_M(f)$: $H_M(XM) \to H_M(YM)$ is not the zero map. So H_M is injective, therefore, bijective. This proves the theorem.

2.22. *Remarks.* Assume (2.6).

(i) Obviously, if $U \leqslant M$, then $H_M(U) = \operatorname{Hom}_T(P, U) = H_P(U)$. By (2.8), $\ker \beta \leqslant \ker \varphi$ for all $\varphi \in \operatorname{Hom}_T(P, M)$, and $\varphi \mapsto \bar{\varphi}$, where $\bar{\varphi}$ is the induced mapping $\bar{\varphi}: P/\ker \beta = M \to M$, is an isomorphism from $\operatorname{Hom}_T(P, M)$ onto $\mathscr{H} = \operatorname{Hom}_T(M, M)$. The $\mathscr{E}$-submodule $\operatorname{Hom}_T(P, U)$ of

$\mathrm{Hom}_T(P, M) \cong \mathrm{Hom}_T(M, M)$ consists of all endomorphisms ψ of M whose image is contained in $U \leqslant M$. So $H_M(U) = \mathrm{Hom}_T(P, U) \cong \mathrm{Hom}_T(M, U)$ canonically. In other words, on submodules of M, the functor H_M acts as the Hom functor $\mathrm{Hom}_T(M, -)$. This, however, is not true in general for arbitrary T-modules U.

(ii) In (2.21) we used for the injectivity part of the proof only the fact that YM is a P-torsionless submodule of M. So H_M acts injectively on $\mathrm{Hom}_T(V, U)$ for arbitralry $V \in \mathrm{mod}\, T$ and P-torsionless submodules U of M.

By functorality H_M preserves composition of mappings. We have the following corollary:

2.23. Corollary. *Assume* (2.6) *and suppose that M is P-torsionless. Let $X_1, ..., X_k$ be right ideals of $\mathscr{H}$ and let $X = \bigoplus_{i=1}^k X_i$, $U = \bigoplus_{i=1}^k X_i M$ (external direct sum). Then H_M induces an R-algebra isomorphism from* $\mathrm{End}_T(U)$ *onto* $\mathrm{End}_{\mathscr{H}}(X)$.

2.24. *Remark.* Assume (2.6). It is immediate from the definitions that H_M preserves epimorphisms. Moreover, if $U, V \in \mathrm{mod}\, T$ satisfy $\mathrm{Hom}_T(P, U) J_\beta = (0) = \mathrm{Hom}_T(P, V) J_\beta$, hence $H_M(U) = \mathrm{Hom}_T(P, U)$, $H_M(V) = \mathrm{Hom}_T(P, V)$, and if $f\colon U \to V$ is injective, then $H_M(f)$ is injective as well. So in particular for arbitrary $f \in \mathrm{Hom}_T(U, V)$ (U, V as above), $H_M(\ker f) = \ker(H(f))$ and $H(\mathrm{im}\, f) = \mathrm{im}(H(f))$. Note in particular that this applies in the situation of (2.23) to submodules of $U = \bigoplus_{i=1}^k X_i M$.

We now turn to irreducible modules.

2.25. Theorem. *Assume* (2.6) *and let $V \in \mathrm{mod}\, T$ be irreducible. Then $H_M(V) = (0)$ or $\mathrm{Hom}_T(P, V) J_\beta = (0)$ and $H_M(V) = \mathrm{Hom}_T(P, V) \neq (0)$ is an irreducible $\mathscr{H}$-module.*

Proof. The theorem holds for H_P, that is $\mathrm{Hom}_T(P, V) = (0)$ or $\mathrm{Hom}_T(P, V)$ is an irreducible $\mathscr{E}$-module by [2, 6.3]. So, if $H_M(V) \neq (0)$, in particular $\mathrm{Hom}_T(P, V) \neq (0)$, and $\mathrm{Hom}_T(P, V) J_\beta \neq \mathrm{Hom}_T(P, V)$, so $\mathrm{Hom}_T(P, V) J_\beta$ is a proper $\mathscr{E}$-submodule of the irreducible $\mathscr{E}$-module $\mathrm{Hom}_T(P, V)$, hence it is zero. So $H_M(V) = \mathrm{Hom}_T(P, V)$ is an irreducible $\mathscr{H}$-module.

Next we prove that a similar statement holds for the inverse direction, that is, if X is an irreducible $\mathscr{H}$-module, then $\tilde{F}_M \neq (0)$, $\tilde{F}_M(X) \cong \tilde{G}_M(X)$, and $\tilde{F}_M(X)$ is an irreducible T-module. We first need:

2.26. Lemma. *Let X be an $\mathscr{H}$-module. Then $\tau_P(X \otimes_{\mathscr{H}} M) = X \otimes_{\mathscr{H}} M$ and $\tau_P(X \otimes_{\mathscr{E}} P) = X \otimes_{\mathscr{E}} P$.*

Proof. Recall that by [1, 20.10], $X \cong X \otimes_{\mathscr{H}} \mathrm{Hom}_T(P, M) \cong \mathrm{Hom}_T(P, X \otimes_{\mathscr{H}} M)$ given explicitly by $x \mapsto \varphi_{x,\beta} \in \mathrm{Hom}_T(P, X \otimes_{\mathscr{H}} M)$ for $x \in X$, where $\varphi_{x,\beta}(u) = x \otimes \beta(u) \in X \otimes_{\mathscr{H}} M$ for $u \in P$. Since $\beta\colon P \to M$ is surjective, the generators $x \otimes m$ ($x \in X$, $m \in M$) of $X \otimes_{\mathscr{H}} M$ all belong to the image of some homomorphism from P to $X \otimes_{\mathscr{H}} M$ (namely of $\varphi_{x,\beta}$), hence are contained in $\tau_P(X \otimes_{\mathscr{H}} M)$. So $\tau_P(X \otimes_{\mathscr{H}} M) = X \otimes_{\mathscr{H}} M$. Since X is also an $\mathscr{E}$-module we may replace M by P to derive $\tau_P(X \otimes_{\mathscr{E}} P) = X \otimes_{\mathscr{E}} P$.

2.27. THEOREM. *Let $X \in \operatorname{mod} \mathscr{H}$ be irreducible. Then $\tilde{F}_M(X) \neq (0)$, $\tilde{F}_M(X) = \tilde{G}_M(X)$, and $\tilde{F}_M(X)$ is an irreducible T-module.*

Proof. By (2.16), $H_M(\tilde{F}_M(X)) = H_M(F_M(X)) \cong X \neq (0)$, so in particular $\tilde{F}_M(X) \neq (0)$. Let $U \leqslant X \otimes_{\mathscr{H}} M = F_M(X)$. Then the $\mathscr{E}$-module $\mathrm{Hom}_T(P, U)$ is canonically embedded into $X \cong \mathrm{Hom}_T(P, X \otimes_{\mathscr{H}} M)$. Since X is irreducible as $\mathscr{H}$- hence as $\mathscr{E}$-module as well, $\mathrm{Hom}_T(P, U) = (0)$, so by Definition (2.11), $U \leqslant t_P(X \otimes_{\mathscr{H}} M)$, or $\mathrm{Hom}_T(P, U) = X \cong \mathrm{Hom}_T(P, X \otimes_{\mathscr{H}} M)$, that is, the image of every homomorphism from P to $X \otimes_{\mathscr{H}} M$ is already contained in $U \leqslant X \otimes_{\mathscr{H}} M$. Consequently by Definition (2.15), $\tau_P(X \otimes_{\mathscr{H}} M) \leqslant U$, if $\mathrm{Hom}_T(P, U) \neq (0)$. By (2.26), $\tau_P(X \otimes_{\mathscr{H}} M) = X \otimes_{\mathscr{H}} M$ forcing $U = X \otimes_{\mathscr{H}} M$. We have shown that all proper submodules of $X \otimes_{\mathscr{H}} M$ are contained in $t_P(X \otimes_{\mathscr{H}} M)$, so this submodule is the unique maximal submodule of $X \otimes_{\mathscr{H}} M$ and the factor module $\tilde{F}_M(X) = (X \otimes_{\mathscr{H}} M)/t_P(X \otimes_{\mathscr{H}} M)$ is irreducible. Since X is also irreducible as $\mathscr{E}$-module, this result applied to the situation where $M = P$ shows that $\tilde{G}_M(X)$ is an irreducible T-module as well. Now $\beta \otimes 1$: $X \otimes_{\mathscr{E}} P \to X \otimes_{\mathscr{E}} M \cong X \otimes_{\mathscr{H}} M$ is an epimorphism, so the induced mapping $A_P(\beta \otimes 1)$ (compare (2.10)) maps $\tilde{G}_M(X)$ onto $\tilde{F}_M(X)$, hence is an isomorphism, since both $\tilde{G}_M(X)$ and $\tilde{F}_M(X)$ are irreducible.

Let $V \in \operatorname{mod} T$ be irreducible and suppose that $X = H_M(V)$ is not the zero module. So $X \in \operatorname{mod} \mathscr{H}$ is irreducible by (2.25). Note in particular that $\mathrm{Hom}_T(P, V) = X$ and $\tau_P(V) = V$ and $t_P(V) = (0)$. The standard homomorphism from $X \otimes_{\mathscr{E}} P = \mathrm{Hom}_T(P, V) \otimes_{\mathscr{E}} P = G_M(X)$ to V given by $f \otimes u \mapsto f(u) \in V$ for $f \in \mathrm{Hom}_T(P, V)$, $u \in P$, is obviously surjective, so (2.27) implies $\tilde{F}_M(X) = \tilde{G}_M(X) \cong V$ canonically. Note that by our general assumptions on T, both rings T and $\mathscr{H}$ have only finitely many non-isomorphic irreducible modules.

2.28. LEMMA. *Assume* (2.6). *Let $\mathscr{I}$, $\mathscr{T}$ be complete sets of non-isomorphic irreducible $\mathscr{H}$- respectively T-modules. Let $\mathscr{T}_H = \{V \in \mathscr{T} \mid H_M(V) \neq (0)\}$. Then H_M induces a bijective correspondence between $\mathscr{T}_H$ and $\mathscr{I}$. The inverse of $H_M\colon \mathscr{T}_H \to \mathscr{I}$ is $\tilde{F}_M\colon \mathscr{I} \to \mathscr{T}_H$. On $\mathscr{I}$, the functors $\tilde{F}_M$ and $\tilde{G}_M$ coincide.*

2.29. LEMMA. *Assume* (2.6). *Let $X \in \operatorname{mod} \mathscr{H}$ and let Y be a maximal* *submodule of X. Let i: $Y \to X$ be the canonical embedding, $F_M(X) = V$, and let $U \leqslant V$ be the image of the T-linear map $i \otimes 1_M: Y \otimes_{\mathscr{H}} M \to X \otimes_{\mathscr{H}} M = V$. Then $t_P(V/U)$ is the unique maximal submodule of V/U and the factor module $A_P(V/U)$ is canonically isomorphic to the irreducible T-module $\tilde{F}_M(X/Y)$.*

Proof. Tensoring the exact sequence $0 \to Y \to X \to X/Y \to 0$ by M yields the exact sequence $Y \otimes_{\mathscr{H}} M \to X \otimes_{\mathscr{H}} M \to X/Y \otimes_{\mathscr{H}} M \to 0$ hence $0 \to U \to V \to X/Y \otimes_{\mathscr{H}} M \to 0$. From this the lemma follows easily using (2.27).

Let $X \in \operatorname{mod} \mathscr{H}$ and suppose $X = X_0 > X_1 > X_2 > \cdots$ is a filtration of X such that $Y_i = X_{i-1}/X_i$ is an irreducible $\mathscr{H}$-module for $i \geqslant 1$. Let $V = F_M(X) = X \otimes_{\mathscr{H}} M$, and let V_i $(i \geqslant 0)$ be the canonical image of $X_i \otimes_{\mathscr{H}} M$ in V. Then (2.29) implies:

2.30. Corollary. *Assume (2.6) and let $X_i \in \operatorname{mod} \mathscr{H}$ $(i \geqslant 0)$, $X = X_0$. Then in the notation above, the factor modules $U_i = V_{i-1}/V_i$ in the induced filtration $V = V_0 > V_1 > V_2 > \cdots$, $V_i = F_M(X_i)$, have the following property: $t_P(U_i)$ is the unique maximal submodule of U_i $(i \geqslant 1)$ and the irreducible T-module $A_P(U_i) = U_i/t_P(U_i)$ is canonically isomorphic to $\tilde{F}_M(Y_i)$ for all $i \geqslant 1$.*

2.31. *Remark.* Obviously in (2.29) and (2.30) we may replace $V = F_M(X)$ by $A_P(V) = \tilde{F}_M(X)$ or V by $G_M(X)$ respectively $\tilde{G}_M(X)$ and similar results hold.

The results above apply in particular to modules with composition series: If X and V are both of finite composition length then the multiplicity of the irreducible $\mathscr{H}$-module $Y \in \mathscr{I}$ (in the notation of (2.28)) as composition factor of X equals the multiplicity of the irreducible T-module $\tilde{F}_M(Y) \in \mathscr{T}_H$ as composition factor of V for $V = F_M(X)$, $\tilde{F}(X)$, $G_M(X)$, or $\tilde{G}_M(X)$.

The case where T is a finite dimensional algebra over some field R is of special interest. Here all T- and $\mathscr{H}$-modules involved, being finitely generated by assumption, have composition series. We list now some consequences for this special case, all of them being easily derived from the previous results:

2.32. *Consequences.* T is now a finite dimensional algebra over some field R. Let $M \in \operatorname{mod} T$ and assume (2.6). By Fitting's lemma (see, e.g., [15, 1.4]) there is a bijective correspondence between the indecomposable direct summands of M and projective indecomposable $\mathscr{H}$-modules; more precisely, if $e_1, ..., e_k$ is a complete set of non-isomorphic primitive idempotents of $\mathscr{H}$ (that is, $e_i\mathscr{H} \not\cong e_j\mathscr{H}$ for $i \neq j$), then $\{e_iM \mid 1 \leqslant i \leqslant k\}$ is a complete set of non-isomorphic indecomposable direct summands of M. By

(2.7) we may assume that $P \to^{\beta} M$ is the projective cover of M. Then J_β is contained in the radical J of $\mathscr{E}$, so in particular J_β is nilpotent. We may lift idempotents from $\mathscr{H}$ to $\mathscr{E}$. Consequently we have a one-by-one correspondence between the indecomposable direct summands of P and those of M (given by restricting β to indecomposable direct summands of P). In particular, the projective cover of any indecomposable direct summand N of M is an indecomposable projective T-module, and therefore N has a unique maximal submodule, the radical $J(N)$ of N. For an arbitrary module V let the radical $J(V)$ of V be the intersection of all maximal submodules of V. Setting $S_i = e_i\mathscr{H}/J(e_i\mathscr{H})$, $\mathscr{I} = \{S_i \mid 1 \leqslant i \leqslant k\}$ is a complete set of non-isomorphic irreducible $\mathscr{H}$-modules. Moreover the irreducible T-module $e_iM/J(e_iM)$ is canonically isomorphic to $\tilde{F}_M(S_i)$. We denote this module by $S(i)$, then $\{S(i) \mid 1 \leqslant i \leqslant k\} = \mathscr{T}_H$ in the notation of (2.28); so $\mathscr{T}_H$ consists precisely of those irreducible T-modules which occur as direct summands in the head $M/J(M)$ of M. For $X \in \operatorname{mod} \mathscr{H}$ let $V = \check{H}_M(X)$, where $\check{H}_M$ is one of the functors F_M, $\tilde{F}_M$, G_M, $\tilde{G}_M$. Then the multiplicity of S_i as composition factor of X equals the multiplicity of $S(i)$ as composition factor of V. Applying this in particular to the projective indecomposable $\mathscr{H}$-module $X = e_i\mathscr{H}$ shows that the Cartan matrix of $\mathscr{H}$ also records the multiplicities of irreducible components of $M/J(M)$ in indedomposable direct summands of M.

3. Cyclic Modules

We now assume that M is a cyclic T-module, that is, $M = mT$ for some $m \in M$, where T is as in the previous section. We need a preliminary result:

3.1. Lemma. *Let $M = mT$, and let J be the annihilator of m in T, $J = \operatorname{ann}_T(m) = \{t \in T \mid mt = 0\}$. Let the subalgebra A of T be the idealizer of J in T, that is, $A = \{r \in T \mid rJ \subseteq J\}$. Then $A/J \cong \mathscr{H} = \operatorname{End}_T(M)$ canonically, given by $r + J \mapsto \varphi_r \in \mathscr{H}$ for $r \in A$, where $\varphi_r(mt) = mrt$ for all $t \in T$.*

Proof. Suppose $mt_1 = mt_2$, t_1, $t_2 \in T$. Then $t_1 - t_2 \in J$, so $r(t_1 - t_2) \in J$ for all $t \in A$ and $mrt_1 = mrt_2$, so φ_r is a well-defined mapping from M to M. It is routine to check that φ_r is indeed in $\mathscr{H}$ and that $r \mapsto \varphi_r$ is an algebra homomorphism from A to $\mathscr{H}$ with kernel J. If $\alpha \in \mathscr{H}$, then $\alpha m = mr_\alpha$ for some $r_\alpha \in T$. Let $x \in J$. Then $0 = \alpha(mx) = m(r_\alpha x)$ so $r_\alpha x \in J$, hence $r_\alpha J \subseteq J$ and $r_\alpha \in A$. Obviously $\varphi_{r_\alpha} = \alpha$, so $r \mapsto \varphi_r$ is surjective and $A/J \cong \mathscr{H}$ canonically.

Assume now (2.6) and let $M = mT$ as above. By (2.7) we may choose $P \xrightarrow{\beta} M$ such that:

3.2. (i) $P = eT$ for some idempotent $e \in T$.

(ii) $\beta(e) = m$.

3.3. *Hypothesis.* Assume (2.6). In addition, let $M = mT$ be cyclic and suppose that $P \xrightarrow{\beta} M$ is chosen as in (3.2).

Assume (3.3). Note that $\mathscr{E} = \mathrm{End}_T(P)$ may be identified (and we do so) with eTe acting by left multiplication on eT. Moreover $P^{\#} = \mathrm{Hom}_T(eT, T) \cong Te$ as left T-module and, if $V \in \mathrm{mod}\ T$, then $V \otimes_T Te \cong Ve$ as eTe-module.

3.4. LEMMA. *Let $M = mT$ be a cyclic T-module and suppose that $P \xrightarrow{\beta} M$ satisfies* (3.2) *for some projective T-module $P = eT$. Let $J = \mathrm{ann}_T(m)$. Then* $\ker \beta = eJ$ *and $me = m$. Moreover* $\ker \beta$ *is invariant under left multiplication by elements of $\mathscr{E} = eTe$ if and only if $Me = \mathscr{H}m$, where $\mathscr{H} = \mathrm{End}_T(M)$.*

Proof. $\ker \beta = eJ$ and $me = m$ are checked easily. Suppose $eTe \ker \beta \subseteq \ker \beta$, that is, $eTeJ \subseteq eJ$. We define a function from eTe to $\mathscr{H}$ mapping $ete \in eTe$ to φ_t, where $\varphi_t m = mte$. As in (3.1) we show that this mapping is a surjective algebra homomorphism. In particular $\varphi m \in Me$ for all $\varphi \in \mathscr{H}$, so $\mathscr{H}m \subseteq Me$.

If $x \in Me$, we find $t \in T$ such that $x = mte = mete = \varphi_t m \in \mathscr{H}m$, since $m = me$, so $Me \subseteq \mathscr{H}m$, and we have shown $Me = \mathscr{H}m$. Now assume that $\mathscr{H}m = Me$, and let $t \in T$, $s \in J$. Then $metes = hms = 0$ for some $h \in \mathscr{H}$, since $mete = mte \in Me = \mathscr{H}m$. So $eTeJ \subseteq J$. Since $e = e^2$ we have indeed $eTeJ = eTeeJ = eTe \ker \beta \subseteq eJ = \ker \beta$.

3.5. *Remark.* Let M and P as in (3.4). Then (3.4) says that (2.6) is equivalent to the condition $Me = \mathscr{H}m$. Moreover, in the notation of (3.1) and assuming $Me = \mathscr{H}m$, we have also shown in (3.4) that $eTe \subseteq A$. Since $e^2 = e$, we have $eTe \subseteq eAe$, hence $eTe = eAe$ ($eAe \subseteq eTe$ being trivial). $J \cap eJe = eJe$ is the kernel of $ete \mapsto \varphi_t \in \mathscr{H}$ ($t \in T$) defined in (3.4), so in particular $eJe = J_\beta$.

3.6. COROLLARY. *Let $M = mT$ be a cyclic T-module, $P \xrightarrow{\beta} M$ as in* (3.2), *and assume that $Me = \mathscr{H}m$. Then $H_M(V) \cong V'e/VeJe$ as $\mathscr{H}$-module canonically, where $J = \mathrm{ann}_T(m)$.*

Proof. Note that $P^{\#} = \mathrm{Hom}_T(eT, T) \cong Te$ as eTe-module, hence by (2.3), $\mathrm{Hom}_T(P, V) \cong V \otimes_T P^{\#} = V \otimes_T Te \cong Ve$. By (3.4) and (3.5), $H_M(V) = \mathrm{Hom}_T(P, V)/\mathrm{Hom}_T(P, V)J_\beta \cong Ve/VeJe$ canonically.

4. Decomposition Numbers

In this section we shall apply the general results of section two to the special situation where T is an order in some semisimple algebra over a field. At the end of Section 2 we have seen that the Cartan numbers of $\mathscr{H}$ (assuming (2.6)) actually record multiplicities of irreducibles as composition factors in indecomposable direct summands of M. Our aim in this section is to establish an even more powerful link between decomposition numbers.

So let $\mathcal{O}$ be a discrete complete valuation ring with quotient field K and residue field F. Let T_K be a semisimple K-algebra and let $T_{\mathcal{O}}$ be an $\mathcal{O}$-order in T_K, that is, $T_{\mathcal{O}}$ is an $\mathcal{O}$-subalgebra of T_K which is free of rank $\dim_K(T_K) = n < \infty$ as $\mathcal{O}$-module. Note that $K \otimes_{\mathcal{O}} T_{\mathcal{O}} \cong T_K$ canonically, and often we identify these two K-algebras. Similarly $F \otimes_{\mathcal{O}} T_{\mathcal{O}} \cong T_{\mathcal{O}}/(\pi) T_{\mathcal{O}}$ canonically, where the unique maximal ideal of $\mathcal{O}$ is generated by $\pi \in \mathcal{O}$ and denoted by (π), and we identify these F-algebras as well. A $T_{\mathcal{O}}$-module $M_{\mathcal{O}}$ which is free of finite rank as $\mathcal{O}$-module is called a $T_{\mathcal{O}}$-lattice and an irreducilbe $T_{\mathcal{O}}$-lattice is a $T_{\mathcal{O}}$-lattice $M_{\mathcal{O}}$ such that the T_K-module $K \otimes_{\mathcal{O}} M_{\mathcal{O}} = M_K$ is irreducible.

We remark that all three algebras T_K, $T_{\mathcal{O}}$, and T_F are semiperfect and satisfy the Krull–Schmidt–Azumaya theorem (see, e.g., [4, 6.12]).

4.1. Definition. Let $M_{\mathcal{O}}$ be a $T_{\mathcal{O}}$-lattice, and $\{M_{\mathcal{O}}^{(1)}, ..., M_{\mathcal{O}}^{(k)}\}$ a complete set of non-isomorphic indecomposable direct summands of $M_{\mathcal{O}}$. Let $\{S^{(1)}, ..., S^{(m)}\}$ be a complete set of non-isomorphic irreducible T_K-modules occurring as irreducible components in the semisimple module $M_K = K \otimes_{\mathcal{O}} M_{\mathcal{O}}$. Then the *decomposition matrix* of $M = M_{\mathcal{O}}$ is the $(m \times k)$-matrix $\mathscr{D}_M = (d_{ij})$, where d_{ij} is the multiplicity of $S^{(i)}$ as irreducible component in $M_K^{(j)} = K \otimes_{\mathcal{O}} M_{\mathcal{O}}^{(j)}$.

Note that $\mathscr{D}_M$ is uniquely determined by M up to the order of the rows and te columns.

4.2. Example. *If $M_{\mathcal{O}}$ is the regular $T_{\mathcal{O}}$-lattice $T_{\mathcal{O}}$*, the set $\{M_{\mathcal{O}}^{(1)}, ..., M_{\mathcal{O}}^{(k)}\}$ is a complete set of non-isomorphic projective indecomposable $T_{\mathcal{O}}$-modules (compare, e.g., [4, Sect. 6]) and $\{S^{(1)}, ..., S^{(m)}\}$ is a complete set of non-isomorphic irreducible T_K-modules. The matrix $\mathscr{D}_{T_{\mathcal{O}}}$ is the classical decomposition matrix of the $\mathcal{O}$-order $T_{\mathcal{O}}$ (compare, e.g., [4, Sect. 6]). We denote this matrix in the following by $\mathscr{D}_T$ or $\mathscr{D}$.

We recall some basic facts on $T_{\mathcal{O}}$-lattices. For proofs we refer the readers to [4, Sect. 6]. First every submodule of a $T_{\mathcal{O}}$-lattice is free of finite rank as $\mathcal{O}$-module, thus is a $T_{\mathcal{O}}$-lattice again. If $M = M_K$ is a finite dimensional T_K-module, and if $N_{\mathcal{O}}$ is a finitely generated $T_{\mathcal{O}}$-submodule of M, then $N_{\mathcal{O}}$

is a $T_{\mathcal{C}}$-lattice. If in addition $K \otimes_{\mathcal{C}} N_{\mathcal{C}} \cong M_K$, or equivalently if the $\mathcal{C}$-rank of $N_{\mathcal{C}}$ equals the K-dimension of M, that is, if $N_{\mathcal{C}}$ contains a K-basis of M_K, we define $N_{\mathcal{C}}$ to be a *full $T_{\mathcal{C}}$-lattice in M*. Obviously any $T_{\mathcal{C}}$-module $M_{\mathcal{C}}$ generated by a K-basis of M is a full $T_{\mathcal{C}}$-lattice in M_K. Moreover, if $M_{\mathcal{C}}$ is a $T_{\mathcal{C}}$-lattice then $M_{\mathcal{C}}$ is a full sublattice in $M_K = K \otimes_{\mathcal{C}} M_{\mathcal{C}}$ (identifying $M_{\mathcal{C}}$ and $1 \otimes_{\mathcal{C}} M_{\mathcal{C}} \leqslant M_K$).

Let $M_{\mathcal{C}}$, $N_{\mathcal{C}}$ be $T_{\mathcal{C}}$-lattices, then $K \otimes_{\mathcal{C}} \operatorname{Hom}_{T_{\mathcal{C}}}(M_{\mathcal{C}}, N_{\mathcal{C}}) \cong \operatorname{Hom}_{T_K}(M_K, N)$ canonically, where $M_K = K \otimes_{\mathcal{C}} M_{\mathcal{C}}$ and $N_K = K \otimes_{\mathcal{C}} N_{\mathcal{C}}$, and the mapping $\varphi \mapsto 1 \otimes \varphi$ from $\operatorname{Hom}_{T_{\mathcal{C}}}(M_{\mathcal{C}}, N_{\mathcal{C}})$ into $\operatorname{Hom}_{T_K}(M_K, N_K)$ is an embedding. Since M_K is a completely reducible module its endomorphism ring $\mathscr{H}_K$ is a finite dimensional semisimple K-algebra. Thus $\mathscr{H}_{\mathcal{C}} = \operatorname{End}_{T_{\mathcal{C}}}(M_{\mathcal{C}})$ is an $\mathcal{C}$-order in $\mathscr{H}_K$, in particular $\mathscr{H}_K \cong K \otimes_{\mathcal{C}} \mathscr{H}_{\mathcal{C}}$ canonically.

As an immediate consequence of Fitting's lemma we have the following.

4.3. THEOREM. *Let $M_{\mathcal{C}}$ be a $T_{\mathcal{C}}$-lattice, $\mathscr{H}_{\mathcal{C}} = \operatorname{End}_{T_{\mathcal{C}}}(M_{\mathcal{C}})$. Let $\{e_1, ..., e_k\}$ be a complete set of non-isomorphic primitive idempotents of $\mathscr{H}_{\mathcal{C}}$, and $\{f_1, ..., f_m\}$ a complete set of non-isomorphic primitive idempotents of $\mathscr{H}_K = K \otimes_{\mathcal{C}} \mathscr{H}_{\mathcal{C}}$. Setting $S^{(i)} = f_i M_K$ $(M_K = K \otimes_{\mathcal{C}} M_{\mathcal{C}})$ and $M_{\mathcal{C}}^{(j)} = e_j M_{\mathcal{C}}$, $\{S^{(i)} \mid 1 \leqslant i \leqslant m\}$ is a full set of non-isomorphic irreducible T_K-modules occurring in M_K as irreducible components, and $\{M_{\mathcal{C}}^{(j)} \mid 1 \leqslant j \leqslant k\}$ is a complete set of non-isomorphic indecomposable direct summands of $M_{\mathcal{C}}$. The multiplicity of $S^{(i)}$ as irreducible component in $M_K^{(j)} = K \otimes_{\mathcal{C}} M_{\mathcal{C}}^{(j)}$ equals the multiplicity of the irreducible $\mathscr{H}_K$-module $f_i \mathscr{H}_K$ in $K \otimes_{\mathcal{C}} e_i \mathscr{H}_{\mathcal{C}}$.*

Recall that two idempotents of a ring are isomorphic if the right ideals generated by them are isomorphic. So $\{e_i \mathscr{H}_{\mathcal{C}} \mid 1 \leqslant i \leqslant k\}$ is a complete set of non-isomorphic projective indecomposable $\mathscr{H}_{\mathcal{C}}$-modules, and (4.3) implies in particular:

4.4. COROLLARY. *Let $M_{\mathcal{C}}$ be a $T_{\mathcal{C}}$-lattice, $\mathscr{H}_{\mathcal{C}} = \operatorname{End}_{T_{\mathcal{C}}}(M_{\mathcal{C}})$. Labelling the indecomposable direct summands of $M_{\mathcal{C}}$ and the irreducible components of $M_K = K \otimes_{\mathcal{C}} M_{\mathcal{C}}$ as in* (4.3), *the decomposition matrix $\mathscr{D}_M$ of $M = M_{\mathcal{C}}$ equals the decomposition $\mathscr{D}_{\mathscr{H}_{\mathcal{C}}}$ of $\mathscr{H}_{\mathcal{C}}$.*

Let $M_{\mathcal{C}}$ be a $T_{\mathcal{C}}$-lattice. Then $M_F = F \otimes_{\mathcal{C}} M_{\mathcal{C}}$ is a T_F-module. We call a T_F-module V *liftable* if $V = M_F$ for a $T_{\mathcal{C}}$-lattice $M_{\mathcal{C}}$. If $M_{\mathcal{C}}$ and $N_{\mathcal{C}}$ are $T_{\mathcal{C}}$-lattices, then $F \otimes_{\mathcal{C}} \operatorname{Hom}_{T_{\mathcal{C}}}(M_{\mathcal{C}}, N_{\mathcal{C}}) \subseteq H_{T_F}(M_F, N_F)$ canonically, and we say $\varphi: M_F \to N_F$ is *liftable* if $\varphi \in F \otimes_{\mathcal{C}} \operatorname{Hom}_{T_{\mathcal{C}}}(M_{\mathcal{C}}, N_{\mathcal{C}})$, and we call $\operatorname{Hom}_{T_F}(M_F, N_F)$ liftable if $\operatorname{Hom}_{T_F}(M_F, N_F) = F \otimes_{\mathcal{C}} \operatorname{Hom}_{T_{\mathcal{C}}}(M_{\mathcal{C}}, N_{\mathcal{C}})$. We list some facts which are proved easily as in the case of group algebras (compare, e.g., [4, Sect. 16]). First any projective T_F-module P_F is liftable, in fact, the $T_{\mathcal{C}}$-lattice $P_{\mathcal{C}}$ such that $PF \cong F \otimes_{\mathcal{C}} P_{\mathcal{C}}$ is projective and it is uniquely determined. If $N_{\mathcal{C}}$ is another $T_{\mathcal{C}}$-lattice, then $\operatorname{Hom}_{T_F}(P_F, N_F)$ is liftable In particular for $\mathscr{E}_R = \mathrm{End}_{T_R}(P_R)$, $R = F$, $\mathscr{O}$, or K, we have $\mathscr{E}_R = R \otimes_{\mathscr{O}} \mathscr{E}_{\mathscr{O}}$.

If $M_{\mathscr{O}}$ is a full $T_{\mathscr{O}}$-lattice in $M_K \in \mathrm{mod}\, T_K$, we call $M_F = F \otimes_{\mathscr{O}} M_{\mathscr{O}}$ a *reduction* of M_K (via $T_{\mathscr{O}}$). In general M_F depends on the choice of $M_{\mathscr{O}}$ in M_K, not however the composition multiplicities of M_F (that is, the multiplicities of the irreducible T_F-modules as composition factors of M_F). If S_F is an irreducible T_F-module, P_F its projective cover, and the projective $T_{\mathscr{O}}$-lattice $P_{\mathscr{O}}$ is chosen such that $P_F = F \otimes_{\mathscr{O}} P_{\mathscr{O}}$, then the multiplicity of an irreducible T_K-module M_K as irreducible component of $K \otimes_{\mathscr{O}} P_{\mathscr{O}}$ equals the composition multiplicity of S_F in a reduction M_F of M_K. So the decomposition matrix $\mathscr{D}$ of $T_{\mathscr{O}}$ also records composition multplicities of reductions of the irreducible T_K-modules. This implies in particular that $\mathscr{D}\mathscr{D}^{\mathrm{t}} = \mathscr{C}$, the Cartan matrix of T_F recording composition multiplicities of projective indecomposable T_F-modules, where $\mathscr{D}^{\mathrm{t}}$ is the transpose of $\mathscr{D}$.

Let $M_{\mathscr{O}}$ be a $T_{\mathscr{O}}$-lattice, $M_R = R \otimes_{\mathscr{O}} M_{\mathscr{O}}$ for $R = K$ or F. Let $P_{\mathscr{O}} \xrightarrow{\beta} M_{\mathscr{O}}$ be a projective presentation of $M_{\mathscr{O}}$, and note that $P_{\mathscr{O}}$ is automatically $\mathscr{O}$-free, since it is finitely generated projective by assumption. For $R = F$, $\mathscr{O}$, or K let $\mathscr{E}_R = \mathrm{End}_{T_R}(P_R)$ and $\mathscr{H}_R = \mathrm{End}_{T_R}(M_R)$ with $P_R = R \otimes_{\mathscr{O}} P_{\mathscr{O}}$. Note that $P_R \xrightarrow{\beta_R} M_R$ ($\beta_R = 1_R \otimes_{\mathscr{O}} \beta_{\mathscr{O}}$, $\beta_{\mathscr{O}} = \beta$) is a projective presentation of M_R for any choice of R, and that $\ker \beta_R = R \otimes_{\mathscr{O}} \ker \beta_{\mathscr{O}}$. This is true for $R = K$ by general theory and follows for $R = F$ from the fact that $M_{\mathscr{O}} \cong P_{\mathscr{O}}/\ker \beta_{\mathscr{O}}$ is free as $\mathscr{O}$-module. We also have $\mathscr{E}_K = K \otimes_{\mathscr{O}} \mathscr{E}_{\mathscr{O}}$, $\mathscr{H}_K = K \otimes_{\mathscr{O}} \mathscr{H}_{\mathscr{O}}$, and $\mathscr{E}_F = F \otimes_{\mathscr{O}} \mathscr{E}_{\mathscr{O}}$, but in general only $F \otimes_{\mathscr{O}} \mathscr{H}_{\mathscr{O}} \subseteq \mathscr{H}_F$.

The next theorem provides a useful test to check in the situation above if (2.6) is satisfied, that is, if $\ker \beta_R$ is invariant under the action (from the left) of $\mathscr{E}_R$. We first need, however, a few basic facts about pure submodules which are well known and may be found in the standard literature. Recall, that an $\mathscr{O}$-submodule U of a finitely generated free $\mathscr{O}$-module M is *pure* in M if and only if one of the following equivalent properties holds:

4.5. (i) $\pi U = U \cap \pi M$.

(ii) M/U is torsion free.

(iii) There exists an $\mathscr{O}$-submodule N of M with $M = U \oplus N$.

(iv) $\dim_K(KU) = \dim_F[(U + \pi M)/\pi M]$.

(v) The mapping $F \otimes_{\mathscr{O}} U$ to $F \otimes_{\mathscr{O}} M$, induced by the canonical embedding of U into M, is injective,

where $\pi \in \mathscr{O}$ is again a generator for the unique maximal ideal of $\mathscr{O}$.

If U is an $\mathscr{O}$-submodule of M, we abbreviate $KU \cap M$ by $\sqrt{U}$. Then

4.6. (i) $\sqrt{U}$ is pure in M.

(ii) If V is a pure submodule of M containing U then $\sqrt{U} \subseteq V$.

(iii) $\sqrt{U} = \{m \in M \mid \pi^r m \in U \text{ for some } 0 \leqslant r \in \mathbb{Z}\}$.

(iv) $\sqrt{U}/U$ is an $\mathcal{O}$-torsion module.

(v) $\sqrt{U} = U$ if and only if U is pure in M.

(vi) If V is a submodule of M containing U such that V/U is an $\mathcal{O}$-torsion module, then $V \leqslant \sqrt{U}$.

We return to the situation where $M_{\mathcal{O}}$ is a $T_{\mathcal{O}}$-lattice with projective presentation $P_{\mathcal{O}} \xrightarrow{\beta_{\mathcal{O}}} M_{\mathcal{O}}$. Since $P_{\mathcal{O}}/\ker \beta_{\mathcal{O}} \cong M_{\mathcal{O}}$ is a free as $\mathcal{O}$-module, $\ker \beta_{\mathcal{O}}$ is pure in $M_{\mathcal{O}}$.

4.7. THEOREM. *Let $M_{\mathcal{O}}$ be a $T_{\mathcal{O}}$-lattice, and $\beta_{\mathcal{O}} : P_{\mathcal{O}} \to M_{\mathcal{O}}$ a projective presentation of $M_{\mathcal{O}}$. Then the following conditions are equivalent*:

(i) *The T_K-modules* $\ker \beta_K$ *and M_K have no irreducible constituent in common.*

(ii) $\mathscr{E}_K \ker \beta_K \subseteq \ker \beta_K$.

(iii) $\mathscr{E}_{\mathcal{O}} \ker \beta_{\mathcal{O}} \subseteq \ker \beta_{\mathcal{O}}$.

Moreover (iii) *implies that* $\mathscr{E}_F \ker \beta_F \subseteq \ker \beta_F$.

Proof. Since T_K is semisimple, $P_K \cong M_K \oplus \ker \beta_K$ and therefore (i) and (ii) are obviously equivalent. By (4.5), $\ker \beta_{\mathcal{O}}$ is pure in $P_{\mathcal{O}}$, so $\ker \beta_{\mathcal{O}} = \sqrt{\ker \beta_{\mathcal{O}}} = \ker \beta_K \cap P_{\mathcal{O}}$ (considering $P_{\mathcal{O}}$ as a subset of P_K). Moreover, $\mathscr{E}_{\mathcal{O}} = \{\varphi \in \mathscr{E}_K \mid \varphi(P_{\mathcal{O}}) \subseteq P_{\mathcal{O}}\}$. One checks easily that (ii) and (iii) are equivalent. Since $\ker \beta_F = F \otimes_{\mathcal{O}} \ker \beta_{\mathcal{O}}$ and $\mathscr{E}_F = F \otimes_{\mathcal{O}} \mathscr{E}_{\mathcal{O}}$, (iii) implies that $\mathscr{E}_F \ker \beta_F \subseteq \ker \beta_F$.

Let $P_{\mathcal{O}} \xrightarrow{\beta_{\mathcal{O}}} M_{\mathcal{O}}$. Then in order to check if (2.6) holds, (4.7) tells us that we have to investigate the semisimple T_K-module P_K, which is often easier. For example, if char $K = 0$ and if $T_{\mathcal{O}}$ is the group algebra of a finite group, this can be done using character theory, in particular the orthogonality relations.

4.8. COROLLARY. *Assume* (2.6) *for* $P_{\mathcal{O}} \xrightarrow{\beta_{\mathcal{O}}} M_{\mathcal{O}}$, *and let* $\mathscr{H}_R = \operatorname{End}_{T_R}(M_R)$ *and* $\mathscr{E}_R = \operatorname{End}_{T_R}(P_R)$, $R = F$ *or* $\mathcal{O}$. *Then the following diagram is commutative with exact rows and columns*:

$$\begin{tikzcd}
0 \arrow[r] & J_{\beta_{\mathcal{O}}} \arrow[r] \arrow[d] & \mathscr{E}_{\mathcal{O}} \arrow[r] \arrow[d] & \mathscr{H}_{\mathcal{O}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J_{\beta_F} \arrow[r] \arrow[d] & \mathscr{E}_F \arrow[r] \arrow[d] & \mathscr{H}_F \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

where $J_{\beta_R} = \mathrm{Hom}_{T_R}(P_R, \ker \beta_R)$ *and the epimorphisms* $\mathscr{E}_R \to \mathscr{H}_R$ *are induced by* β_R. *In particular* $\mathscr{H}_F = F \otimes_{\mathcal{C}} \mathscr{H}_{\mathcal{C}} \cong \mathscr{H}_{\mathcal{C}}/(\pi)\mathscr{H}_{\mathcal{C}}$. *Moreover, if* $\{M_{\mathcal{C}}^{(1)}, ..., M_{\mathcal{C}}^{(k)}\}$ *is a complete set of indecomposable direct summands of the* $T_{\mathcal{C}}$*-module* $M_{\mathcal{C}}$, *then* $\{M_F^{(1)}, ..., M_F^{(k)}\}$, *where* $M_F^{(i)} = F \otimes_{\mathcal{C}} M_{\mathcal{C}}^{(i)}$ $(1 \leqslant i \leqslant k)$, *is a complete set of indecomposable direct summands of the* T_F*-module* M_F. *In particular every direct summand of* M_F *is liftable. Finally, setting* $P_R^{(i)} = \beta_R^{-1}(M_F^{(i)})$, $R = F$ *or* $\mathcal{C}$, $1 \leqslant i \leqslant k$, $P_R^{(i)} \xrightarrow{\beta_R} M_R^{(i)}$ *is the projective cover of* $M_R^{(i)}$, *the* T_R*-modules* $P_R^{(i)}$ *are projective indecomposable* (*so* $M_F^{(i)}$ *has simple head*), *and* $P_R = P_R^{(1)} \oplus \cdots \oplus P_R^{(k)} \oplus Q_R$, *where* Q_R *is projective contained in* $\ker \beta_R$ *and* $Q_F = F \otimes_{\mathcal{C}} Q_{\mathcal{C}}$.

Proof. Theorem (4.7) implies that $P_F \xrightarrow{\beta_F} M_F$ satisfies (2.6). So $\mathscr{H}_F = \mathscr{E}_F / J_{\beta_F}$. Now $J_{\beta_F} = F \otimes_{\mathcal{C}} \mathrm{Hom}_{T_{\mathcal{C}}}(P_{\mathcal{C}}, \ker \beta_{\mathcal{C}}) = F \otimes_{\mathcal{C}} J_{\beta_{\mathcal{C}}}$, since $J_{\beta_F} = \mathrm{Hom}_{T_F}(P_F, \ker \beta_F)$ is liftable by projectivity of P_F. So $J_{\beta_{\mathcal{C}}} \to J_{\beta_F}$ and similarly $\mathscr{E}_{\mathcal{C}} \to \mathscr{E}_F$ are surjective. It is immediate from the definition of the mappings involved that the diagram is commutative. So the canonical mapping $\mathscr{H}_{\mathcal{C}} \to \mathscr{H}_F$ must be surjective as well. The other claims now follow easily using (2.32) and lifting idempotents (compare, for example, [4, 6.7]).

We are now prepared to link the decomposition matrix of $T_{\mathcal{C}}$ and $\mathscr{H}_{\mathcal{C}}$ (under hypothesis (2.6)).

4.9. THEOREM. *Assume that* (2.6) *holds for* $P_{\mathcal{C}} \xrightarrow{\beta_{\mathcal{C}}} M_{\mathcal{C}}$. *Let* $M_R = R \otimes_{\mathcal{C}} M_{\mathcal{C}}$, $\mathscr{H}_R = \mathrm{End}_{T_R}(M_R)$, $R = F$, $\mathcal{C}$, *or* K, *and let* $e \in \mathscr{H}_{\mathcal{C}}$, $f \in \mathscr{H}_K$ *be primitive idempotents. Set* $N_{\mathcal{C}} = eM_{\mathcal{C}}$, $S_K = fM_K$, *and let* $Q_{\mathcal{C}}$ *be the projective cover of* $N_{\mathcal{C}}$. *Then* $Q_{\mathcal{C}}$ *is indecomposable and* S_K *is irreducible. Moreover the multiplicity of* S_K *as irreducible component in* $Q_K = K \otimes_{\mathcal{C}} Q_{\mathcal{C}}$ *equals the multiplicity of the irreducible* $\mathscr{H}_K$*-module* $f\mathscr{H}_K$ *in* $e\mathscr{H}_K$.

Proof. Note that $\mathscr{H}_K$ is semisimple since M_K is completely reducible. So $e\mathscr{H}_{\mathcal{C}}$ is a projective indecomposable $\mathscr{H}_{\mathcal{C}}$-module and $f\mathscr{H}_K$ is irreducible. Consequently the multiplicity of $f\mathscr{H}_K$ in $e\mathscr{H}_K = K \otimes_{\mathcal{C}} e\mathscr{H}_{\mathcal{C}}$ is a decomposition number of $\mathscr{H}_{\mathcal{C}}$. By Fitting's lemma S_K is irreducible and $N_{\mathcal{C}}$ is an indecomposable direct summand of $M_{\mathcal{C}}$. By (4.8) we may assume that $Q_{\mathcal{C}}$ is the indecomposable direct summand $\beta_{\mathcal{C}}^{-1}(N_{\mathcal{C}})$ of $M_{\mathcal{C}}$, so $\gamma_{\mathcal{C}} : Q_{\mathcal{C}} \to N_{\mathcal{C}}$ is an epimorpism, $\gamma_{\mathcal{C}}$ being the restriction of $\beta_{\mathcal{C}}$ to $Q_{\mathcal{C}}$. In particular, by (4.7)(i), $\ker \gamma K \subseteq \ker \beta_K$, for $\gamma_K = 1_K \otimes_{\mathcal{C}} \gamma_{\mathcal{C}}$, and $N_K \leqslant M_K$ imply that $\ker \gamma_K$ and N_K have no irreducible component in common. Since $S_K \leqslant M_K$, again by (4.7)(i), S_K is not a component of $\ker \gamma_K$. Since $Q_K = \ker \gamma_K \oplus N_K$, the multiplicity of S_K as irreducible component in Q_K equals the multiplicity of S_K as irreducible component of N_K. This multiplicity, however, equals the multiplicity of $f\mathscr{H}_K$ as irreducible component in $e\mathscr{H}_K$ by (4.3).

Recall that by (2.7) we may assume in (4.9) that $P_{\mathcal{C}} \xrightarrow{\beta_{\mathcal{C}}} M_{\mathcal{C}}$ is a projective cover of $M_{\mathcal{C}}$. As a consequence, by general theory, no direct summand of $P_{\mathcal{C}}$ is contained in the kernel of $\beta_{\mathcal{C}}$, and (4.8) says in particular that there is a canonical bijection between the isomorphism classes of indecomposable direct summands of $M_{\mathcal{C}}$ and the non-isomorphic projective indecomposable direct summands of $P_{\mathcal{C}}$. So using Fitting's theorem we get a canonical one-by-one correspondence between those columns of the decomposition matrix $\mathscr{D}_T = \mathscr{D}_{T_{\mathcal{C}}}$ of $T_{\mathcal{C}}$ associated with indecomposable direct summands of $P_{\mathcal{C}}$ and the columns of the decomposition matrix $\mathscr{D}_{\mathscr{H}} = \mathscr{D}_{\mathscr{H}_{\mathcal{C}}}$ which equals $\mathscr{D}_M$ by (4.4), and similarly a canonical bijection between the rows of $\mathscr{D}_T$ corresponding to irreducible components of M_K and the rows of $\mathscr{D}_{\mathscr{H}}$. With this identification of column and row labels we have shown:

4.10. COROLLARY. *Assume* (2.6) *for* $P_{\mathcal{C}} \xrightarrow{\beta_{\mathcal{C}}} M_{\mathcal{C}}$. *Then the decomposition matrix* $\mathscr{D}_{\mathscr{H}}$ of $\mathscr{H}_{\mathcal{C}} = \operatorname{End}_{T_{\mathcal{C}}}(M_{\mathcal{C}})$ *is a submatrix of the decomposition matrix* $\mathscr{D}_T$ of $T_{\mathcal{C}}$.

4.11. *Remark.* Suppose $P_{\mathcal{C}} \xrightarrow{\beta_{\mathcal{C}}} M_{\mathcal{C}}$ is a projective cover of $M_{\mathcal{C}}$. In (2.32) we defined a one-by-one correspondence between the non-isomorphic irreducible T_F-modules in the head of M_F and the irreducible $\mathscr{H}_F$-modules. Let $f \in \mathscr{H}_K$ be a primitive idempotent, and define $S_{\mathcal{C}}$ to be the pure sublattice $fM_K \cap M_{\mathcal{C}}$ of $M_{\mathcal{C}}$. Then the row of $\mathscr{D}_{\mathscr{H}}$ corresponding to $f\mathscr{H}_K$ records the multiplicities of the irreducible T_F-modules occurring in the head of M_F in the reduction $S_F = F \otimes_{\mathcal{C}} S_{\mathcal{C}}$ of the irreducible T_K-module $S_K = fM_K$. This, however, can be directly deduced from the discussion preceding (2.32). We actually get more, namely (2.32) in particular relates composition series of $F \otimes_{\mathcal{C}} (f\mathscr{H}_K \cap \mathscr{H}_{\mathcal{C}})$ with certain filtrations of S_F. Note that the columns of $\mathscr{D}_{\mathscr{H}}$ describe multiplicities of the irreducible T_K-modules occurring in M_K as irreducible components in T_K-modules, which arise by tensoring the principal indecomposable direct summands of $P_{\mathcal{C}}$ by K. In general we get neither complete rows nor columns of $\mathscr{D}_T$.

For the remainder of this section we assume always that $P_{\mathcal{C}} \xrightarrow{\beta_{\mathcal{C}}} M_{\mathcal{C}}$ satisfies (2.6). We are going to extend (2.17) and (2.21) to a larger class of modules. Remark that we are now confronted with three functors, namely H_{M_R}: mod $T_R \to$ mod $\mathscr{H}_R$ for $R = F$, $\mathcal{C}$, or K. Henceforth we will write H_R instead of H_{M_R}.

4.12. LEMMA. *Let* $U_{\mathcal{C}}$ *and* $V_{\mathcal{C}}$ *be* $T_{\mathcal{C}}$*-lattices, and let* $\varphi: U_{\mathcal{C}} \to V_{\mathcal{C}}$ *be* $T_{\mathcal{C}}$*-linear. As usual let* $J_{\beta_{\mathcal{C}}} = \operatorname{Hom}_{T_{\mathcal{C}}}(P_{\mathcal{C}}, \ker \beta_{\mathcal{C}})$. *Suppose* $\operatorname{Hom}_{T_{\mathcal{C}}}(P_{\mathcal{C}}, U_{\mathcal{C}}) J_{\beta_{\mathcal{C}}} = (0) = \operatorname{Hom}_{T_{\mathcal{C}}}(P_{\mathcal{C}}, V_{\mathcal{C}}) J_{\beta_{\mathcal{C}}}$. *Then*:

(i) $\mathscr{H}_{\mathcal{C}}(U_{\mathcal{C}})$ *is free as* $\mathcal{C}$*-module, and* $H_R(U_R) \cong R \otimes_{\mathcal{C}} H_{\mathcal{C}}(U_{\mathcal{C}})$ *for* $R = F$ *or* K.

(ii) *The following diagram is commutative, the vertical mappings being the canonical epimorphisms* $\alpha \mapsto 1_R \otimes_{\mathscr{C}} \alpha$ *for* $\alpha \in H_{\mathscr{C}}(U_{\mathscr{C}})$ *respectively* $\alpha \in H_0(V_0)$:

$$\begin{array}{ccc} H_{\mathscr{C}}(U_{\mathscr{C}}) & \xrightarrow{H_{\mathscr{C}}(\varphi_{\mathscr{C}})} & H_{\mathscr{C}}(V_{\mathscr{C}}) \\ \downarrow & & \downarrow \\ H_R(U_R) & \xrightarrow{H_R(\varphi_R)} & H_R(V_R) \end{array}$$

for $R = K$ *or* F.

Proof. Since $\mathrm{Hom}_{T_{\mathscr{C}}}(P_{\mathscr{C}}, U_{\mathscr{C}})J_{\beta_{\mathscr{C}}} = (0)$, we have $\mathrm{Hom}_{T_R}(P_R, U_R)J_{\beta_R} = (0)$, therefore $H_R(U_R) = \mathrm{Hom}_{T_R}(P_R, U_R)$ for $R = F$, $\mathscr{C}$, or K. Similarly $H_R(V_R) = \mathrm{Hom}_{T_R}(P_R, V_R)$. Now (i) follows immediately for $R = K$ and from the fact that $P_{\mathscr{C}}$ is projective for $R = F$.

To prove (ii) let $\alpha \in H_{\mathscr{C}}(U_{\mathscr{C}})$, that is, let α be a $T_{\mathscr{C}}$-linear mapping from $P_{\mathscr{C}}$ to $U_{\mathscr{C}}$. Then $1_R \otimes_{\mathscr{C}} (H_{\mathscr{C}}(\varphi_{\mathscr{C}}))(\alpha) = 1_R \otimes_{\mathscr{C}} (\varphi_{\mathscr{C}}\alpha) = 1_R \otimes_{\mathscr{C}} \varphi_{\mathscr{C}})(1_R \otimes_{\mathscr{C}} \alpha) = (H_R(\varphi_R))(1_R \otimes_{\mathscr{C}} \alpha)$, so (ii) holds.

We remark that $\mathrm{Hom}_{T_R}(P_R, V_R)J_{\beta_R} = (0)$ ($R = F$, $\mathscr{C}$, or K) for every submodule or epimorphic image V_R of any T_R-module U_R for which $\mathrm{Hom}_{T_R}(P_R, U_R)J_{\beta_R} = (0)$. This applies in particular to modules of the form $X_R M_R$, where X_R is a subset of $\mathscr{H}_R$. Recall Definition (4.15) of the P_R-trace $\tau_{P_R}(V)$ of a T_R-module V.

4.13. Lemma. *Let* X *be a right ideal of* $\mathscr{H}_{\mathscr{C}}$, *and denote* $\tau_{P_{\mathscr{C}}}$ *by* $\tau_{\mathscr{C}}$. *Then* $\tau_{\mathscr{C}}(\sqrt{XM_{\mathscr{C}}}) = \tau_{\mathscr{C}}(\sqrt{X}\, M_{\mathscr{C}})$, *where* $\sqrt{XM_{\mathscr{C}}} = X_K M_K \cap M_{\mathscr{C}}$ *and* $\sqrt{X} = X_K \cap \mathscr{H}_{\mathscr{C}}$, $X_K = K \otimes_{\mathscr{C}} X$. *In particular if* X *is pure in* $\mathscr{H}_{\mathscr{C}}$, *then* $\tau_{\mathscr{C}}$, *then* $\tau_{\mathscr{C}}(\sqrt{XM_{\mathscr{C}}}) = \tau_{\mathscr{C}}(XM_{\mathscr{C}})$.

Proof. Obviously $XM_{\mathscr{C}} \subseteq \sqrt{X}\, M_{\mathscr{C}}$. Let $m \in \sqrt{X}\, M_{\mathscr{C}}$. Then we may write m as a finite sum of terms $\alpha_i m_i$ with $m_i \in M_{\mathscr{C}}$, $\alpha_i \in \sqrt{X}$. So we find $k \in \mathbb{N}$ such that $\pi^k \alpha_i \in X$ for all i, hence $\pi^k m \in XM_{\mathscr{C}}$ and therefore $m \in \sqrt{XM_{\mathscr{C}}}$. We have shown $\sqrt{X}\, M_{\mathscr{C}} \subseteq \sqrt{XM_{\mathscr{C}}}$. So $\sqrt{XM_{\mathscr{C}}} = \sqrt{\sqrt{X}\, M_{\mathscr{C}}}$ and we may assume that $\sqrt{X} = X$ is pure in $\mathscr{H}_{\mathscr{C}}$.

Let $\alpha\colon P_{\mathscr{C}} \to \sqrt{XM_{\mathscr{C}}}$ be $T_{\mathscr{C}}$-linear. Then, since P is finitely generated, we find $k \in \mathbb{N}$ such that $\pi^k \alpha(P) \subseteq XM_{\mathscr{C}}$. So embedding $\mathrm{Hom}_{T_{\mathscr{C}}}(P_{\mathscr{C}}, XM_{\mathscr{C}})$ into $\mathrm{Hom}_{T_{\mathscr{C}}}(P_{\mathscr{C}}, \sqrt{XM_{\mathscr{C}}})$ and then $\mathrm{Hom}_{T_{\mathscr{C}}}(P_{\mathscr{C}}, \sqrt{XM_{\mathscr{C}}})$ into $\mathrm{Hom}_{T_{\mathscr{C}}}(P_{\mathscr{C}}, M_{\mathscr{C}}) \cong \mathscr{H}_{\mathscr{C}}$ using the embeddings $XM_{\mathscr{C}} \to \sqrt{XM_{\mathscr{C}}} \to M_{\mathscr{C}}$, we have shown that $\mathrm{Hom}_{T_{\mathscr{C}}}(P_{\mathscr{C}}, \sqrt{XM_{\mathscr{C}}}) \subseteq \sqrt{\mathrm{Hom}_{T_{\mathscr{C}}}(P_{\mathscr{C}}, XM_{\mathscr{C}})}$. By (2.20) the canonical mapping $\mathrm{Hom}_{T_{\mathscr{C}}}(P_{\mathscr{C}}, X \otimes_{\mathscr{H}_{\mathscr{C}}} M_{\mathscr{C}}) \to \mathrm{Hom}_{T_{\mathscr{C}}}(P_{\mathscr{C}}, XM_{\mathscr{C}})$ is an isomorphism. By (2.16) we have an isomorphism, naturally in X, from $\mathrm{Hom}_{T_{\mathscr{C}}}(P_{\mathscr{C}}, X \otimes_{\mathscr{H}_{\mathscr{C}}} M_{\mathscr{C}})$ onto X. As a consequence the following diagram is commutative, the vertical mappings being bijective, the horizontal ones injective:

$$\begin{array}{ccc} \mathrm{Hom}_{T_{\mathcal{C}}}(P_{\mathcal{C}}, XM_{\mathcal{C}}) & \longrightarrow & \mathrm{Hom}_{T_{\mathcal{C}}}(P_{\mathcal{C}}, M_{\mathcal{C}}) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathscr{H}_{\mathcal{C}} \end{array}$$

In particular $\mathrm{Hom}_{T_{\mathcal{C}}}(P_{\mathcal{C}}, XM_{\mathcal{C}})$ is pure in $\mathrm{Hom}_{T}(P_{\mathcal{C}}, M_{\mathcal{C}})$, since X is pure in $\mathscr{H}_{\mathcal{C}}$ by assumption. So $\mathrm{Hom}_{T_{\mathcal{C}}}(P_{\mathcal{C}}, \sqrt{XM_{\mathcal{C}}})$ is contained in hence equal to $\mathrm{Hom}_{T_{\mathcal{C}}}(P_{\mathcal{C}}, XM_{\mathcal{C}})$. As as consequence if $\alpha: P_{\mathcal{C}} \to \sqrt{XM_{\mathcal{C}}}$ is $T_{\mathcal{C}}$-linear, $\mathrm{im}\, \alpha \subseteq XM_{\mathcal{C}}$. So $\tau_{\mathcal{C}}(XM_{\mathcal{C}}) = \tau_{\mathcal{C}}(\sqrt{XM_{\mathcal{C}}})$.

4.14. COROLLARY. *Let X be a right ideal of $\mathscr{H}_{\mathcal{C}}$. Then $H_{\mathcal{C}}(\sqrt{XM_{\mathcal{C}}}) = \sqrt{X}$. In particular, if $X = X_K \cap \mathscr{H}_{\mathcal{C}}$, then $H_{\mathcal{C}}(\sqrt{XM_{\mathcal{C}}}) = X$.*

Proof. Using (2.17)(i), (4.13), and (2.20) we have $H_{\mathcal{C}}(\sqrt{XM_{\mathcal{C}}}) = H_{\mathcal{C}}(\tau_{\mathcal{C}}\sqrt{XM_{\mathcal{C}}}) = H_{\mathcal{C}}(\sqrt{X}\, M_{\mathcal{C}}) = \sqrt{X}$.

4.15. COROLLARY. *Let $X = X_{\mathcal{C}}$ be a right ideal of $\mathscr{H}_{\mathcal{C}}$ such that $\sqrt{X} = X$. Then $H_F(F \otimes_{\mathcal{C}} X_{\mathcal{C}} M_{\mathcal{C}}) \cong H_F(X_F M_F) \cong H_F(F \otimes_{\mathcal{C}} \sqrt{XM_{\mathcal{C}}}) \cong X_F$ as $\mathscr{H}_F$-module, where $X_F = F \otimes_{\mathcal{C}} X_{\mathcal{C}}$.*

Proof. By projectivity of P_F and (4.14), $\mathrm{Hom}_{T_F}(P_F, F \otimes_{\mathcal{C}} X_{\mathcal{C}} M_{\mathcal{C}}) \cong F \otimes_{\mathcal{C}} \mathrm{Hom}_{T_{\mathcal{C}}}(P_{\mathcal{C}}, X_{\mathcal{C}} M_{\mathcal{C}}) \cong F \otimes_{\mathcal{C}} X_{\mathcal{C}} = X_F$. In particular $\mathrm{Hom}_{T_F}(P_F, F \otimes_{\mathcal{C}} X_{\mathcal{C}} M_{\mathcal{C}}) J_{\beta_F} = (0)$ and therefore $H_F(F \otimes_{\mathcal{C}} X_{\mathcal{C}} M_{\mathcal{C}}) = \mathrm{Hom}_{T_F}(P_F, F \otimes_{\mathcal{C}} X_{\mathcal{C}} M_{\mathcal{C}}) \cong X_F$. Similarly using (4.14), $H_F(F \otimes_{\mathcal{C}} \sqrt{XM_{\mathcal{C}}}) \cong X_F$. By (2.20), $H_F(X_F M_F) = X_F$.

4.16. *Remark.* The canonical epimorphism $F \otimes X_{\mathcal{C}} M_{\mathcal{C}} \to X_F M_F$ induces a canonical epimorphism $\mathrm{Hom}_{T_F}(P_F, F \otimes X_{\mathcal{C}} M_{\mathcal{C}}) \to \mathrm{Hom}_{T_F}(P_F, X_F M_F)$, and similarly the canonical embedding $X_F M_F \to F \otimes \sqrt{X_{\mathcal{C}} M_{\mathcal{C}}})$ induces an embedding of $\mathrm{Hom}_{T_F}(P_F, X_F M_F)$ into $\mathrm{Hom}_{T_F}(P_F, F \otimes_{\mathcal{C}} \sqrt{X_{\mathcal{C}} M_{\mathcal{C}}})$ for arbitrary right ideals $X_{\mathcal{C}}$ of $\mathscr{H}_{\mathcal{C}}$. If $X_{\mathcal{C}}$ is pure in $\mathscr{H}_{\mathcal{C}}$, then (4.15) states that these mappings of Hom-sets are isomorphisms.

Note also that (4.15) implies in particular (for pure X) that $\tau_F(F \otimes_{\mathcal{C}} \sqrt{X_{\mathcal{C}} M_{\mathcal{C}}}) = \tau_F(X_F M_F)$. Moreover the kernel of the canonical epimorphism $F \otimes_{\mathcal{C}} X_{\mathcal{C}} M_{\mathcal{C}} \to X_F M_F$ is P_F-torsion free, that is, $A_{P_F}(F \otimes_{\mathcal{C}} X_{\mathcal{C}} M_{\mathcal{C}}) = A_{P_F}(X_F M_F)$. Obviously we may extract statements from (4.15) also if we drop the assumption that $X_{\mathcal{C}}$ is pure in $\mathscr{H}_{\mathcal{C}}$: Thus $H_F(F \otimes_{\mathcal{C}} X_{\mathcal{C}} M_{\mathcal{C}}) = F \otimes_{\mathcal{C}} X_{\mathcal{C}}$ and $H_F(F \otimes_{\mathcal{C}} \sqrt{X_{\mathcal{C}} M_{\mathcal{C}}}) = F \otimes_{\mathcal{C}} \sqrt{X}$. However, since $X_F = F \otimes_{\mathcal{C}} X_{\mathcal{C}}$ is no longer a subset of $\mathscr{H}_F$, $X_F M_F$ is not defined. However, we could set X_F to be $(X_{\mathcal{C}} + (\pi)\mathscr{H}_{\mathcal{C}})/(\pi)\mathscr{H}_{\mathcal{C}}$ then $H_F(X_F M_F) = X_F$. Similarly the following theorem could be extended to non-pure right ideals of $\mathscr{H}_{\mathscr{C}}$. However, we shall use in the upcoming applications only pure right ideals.

4.17. THEOREM. *Suppose that $M_{\mathscr{C}}$ is $P_{\mathscr{C}}$-torsionless. Let X, Y be pure right ideals of $\mathscr{H}_{\mathscr{C}} = \operatorname{End}_{T_{\mathscr{C}}}(M_{\mathscr{C}})$. Let $R = F$, $\mathscr{C}$, or K, then the functor H_R induces an isomorphism, also denoted by H_R:*

$$H_R\colon \operatorname{Hom}_{T_R}(R \otimes_{\mathscr{C}} \sqrt{X_{\mathscr{C}} M_{\mathscr{C}}}, R \otimes_{\mathscr{C}} \sqrt{Y_{\mathscr{C}} M_{\mathscr{C}}}) \\ \to \operatorname{Hom}_{\mathscr{H}_{\mathscr{A}}}(R \otimes_{\mathscr{C}} X_{\mathscr{C}}, R \otimes_{\mathscr{C}} Y_{\mathscr{C}}).$$

Moreover $H_R = 1_R \otimes_{\mathscr{C}} H_{\mathscr{C}}$ canonically.

Proof. Note that M_R is P_R-torsionless for all choices of R. So H_R is an injective R-linear map by (2.21). Let $\varphi\colon X_{\mathscr{C}} \to Y_{\mathscr{C}}$. By (2.21), $\varphi = H_{\mathscr{C}}(\psi)$ for some $T_{\mathscr{C}}$-linear $\psi\colon X_{\mathscr{C}} M_{\mathscr{C}} \to Y_{\mathscr{C}} M_{\mathscr{C}}$. Let ψ_K denote the T_K-linear map $1 \otimes \psi\colon X_K M_K \to Y_K M_K$ extended to an endomorphism of M_K. Let $m \in \sqrt{X_{\mathscr{C}} M_{\mathscr{C}}}$, then $\pi^k m \in X_{\mathscr{C}} M_{\mathscr{C}}$ for some natural number k. So $\pi^k(\psi_K(m)) = \psi_K(\pi^k m) = \psi(\pi^k(m))$ is an element of $Y_{\mathscr{C}} M_{\mathscr{C}} \subseteq M_K$, so $\psi_K(m) \in \sqrt{Y_{\mathscr{C}} M_{\mathscr{C}}}$. We have shown that $\psi\colon X_{\mathscr{C}} M_{\mathscr{C}} \to Y_{\mathscr{C}} M_{\mathscr{C}}$ can be extended to a $T_{\mathscr{C}}$-linear map $\psi_K\colon \sqrt{X_{\mathscr{C}} M_{\mathscr{C}}} \to \sqrt{Y_{\mathscr{C}} M_{\mathscr{C}}}$ (i.e., the restriction of ψ_K to $\sqrt{X_{\mathscr{C}} M_{\mathscr{C}}}$).

Now $H_{\mathscr{C}}(\psi_K)(\alpha) = \psi_K \alpha = \psi_{\mathscr{C}} \alpha = H_{\mathscr{C}}(\psi_{\mathscr{C}}) = \varphi$ for $\alpha\colon P_{\mathscr{C}} \to \sqrt{X_{\mathscr{C}} M_{\mathscr{C}}}$, since by (4.13), $\operatorname{im} \alpha \subseteq X_{\mathscr{C}} M_{\mathscr{C}}$. This shows that $H_{\mathscr{C}}$ maps $\operatorname{Hom}_{T_{\mathscr{C}}}(\sqrt{X_{\mathscr{C}} M_{\mathscr{C}}}, \sqrt{Y_{\mathscr{C}} M_{\mathscr{C}}})$ onto $\operatorname{Hom}_{\mathscr{H}_{\mathscr{A}}}(X_{\mathscr{C}}, Y_{\mathscr{C}})$, therefore is an isomorphism. This proves the theorem for $R = \mathscr{C}$. The cases $R = K$ and F follow from (4.12).

In the sequel to this paper we shall apply the general theory developed here to the representation theory of general linear groups. The main idea will be based upon the following procedure:

Step 1. We define a certain $T_{\mathscr{C}}$-lattice $M_{\mathscr{C}}$ and verify (2.6) by showing that (4.7)(i) holds. (This will actually be done by using (3.4).) The endomorphism ring of $M_R = R \otimes_{\mathscr{C}} M_{\mathscr{C}}$ is $\mathscr{H}_R$ ($R = F$, $\mathscr{C}$, or K).

Step 2. We take certain pure right ideals $X_1, \ldots, X_k$ of $\mathscr{H}_{\mathscr{C}}$, and define the $\mathscr{H}_R$-module X_R to be

$$X_R = R \otimes_{\mathscr{C}} \bigoplus_{i=1}^{k} X_i,$$

and the T_R-module N_R to be

$$N_R = \bigoplus_{i=1}^{k} R \otimes_{\mathscr{C}} \sqrt{X_i M_{\mathscr{C}}}.$$

Then (4.17) (compare (2.23)) guarantees that

4.18. $\mathcal{S}_R = \mathrm{End}_{T_R}(N_R) \cong R \otimes_{\mathcal{C}} \mathrm{End}_{T_{\mathcal{C}}}(N_{\mathcal{C}}) \cong R \otimes_{\mathcal{C}} \mathrm{End}_{\mathcal{H}_{\mathcal{C}}}(X_{\mathcal{C}}) \cong \mathrm{End}_{\mathcal{H}_R}(X_R)$.

Step 3. We verify (2.6) by showing that (4.7)(i) holds, now for the $T_{\mathcal{C}}$-lattice $N_{\mathcal{C}}$.

Lemma (2.28) gives a procedure for using irreducible $\mathcal{S}_F$-modules to define irreducible T_F-modules, and (4.10) connects the decomposition matrices of $T_{\mathcal{C}}$ and $\mathcal{S}_{\mathcal{C}}$. Since $\mathcal{S}_R$ is also the endomorphism ring of an $\mathcal{H}_R$-module (namely X_R) we might use facts we know about $\mathcal{H}_R$ to get information on $\mathcal{S}_R$. Actually in practice one of the direct summands of $X_{\mathcal{C}}$, say X_1, will be the regular $\mathcal{H}_{\mathcal{C}}$-module $\mathcal{H}_{\mathcal{C}}$. So, if $e: X_{\mathcal{C}} \to X_1$ is the projection onto X_1, then $\mathcal{H}_R \cong e\mathcal{S}_R e$ canonically, and the projective $\mathcal{S}_R$-module $e\mathcal{S}_R$ satisfies (2.6) trivially. We get the commutative diagram of functors

$$\begin{tikzcd} \mathrm{mod}\ T_R \arrow[rr, "H_M"] \arrow[dr, "H_N"'] & & \mathrm{mod}\ \mathcal{H}_R \\ & \mathrm{mod}\ \mathcal{S}_R \arrow[ur, "H_{e\mathcal{S}_R} = \mathrm{Hom}_{\mathcal{S}_R}(e\mathcal{S}_R,\ )"'] & \end{tikzcd} \tag{4.19}$$

linking the representation theory of the three algebras T_R, $\mathcal{H}_R$, and $\mathcal{S}_R$.

ACKNOWLEDGMENTS

I am indebted to Reinhard Knörr for many fruitful discussions on the material in this paper.

REFERENCES

1. F. W. ANDERSON AND K. R. FULLER, "Rings and Categories of Modules," Springer-Verlag Berlin/Heidelberg/New York, 1974).
2. M. AUSLANDER, Representation theory of Artin algebras, I, *Comm. Algebra* **1** (1974), 177–268.
3. M. AUSLANDER, Representation theory of Artin algebras, II, *Comm. Algebra* **1** (1974), 269–310.
4. C. W. CURTIS AND I. REINER, "Methods of Representation Theory with Applications to Finite Groups and Orders," Vol. I, Wiley, New York, 1981.
5. R. DIPPER, On the decomposition numbers of the finite general linear groups, *Trans. Amer. Math. Soc.* **290** (1985), 315–343.
6. R. DIPPER, On the decomposition numbers of the finite general linear groups, II, *Trans. Amer. Math. Soc.* **292** (1985), 123–133.
7. R. DIPPER AND G. D. JAMES, Identification of the irreducible modules of $GL_n(q)$, *J. Algebra* **104** (1986), 266–288.
8. R. DIPPER AND G. D. JAMES, The q-Schur algebra, *Proc. London Math. Soc.*, in press.
9. J. A. GREEN, "Polynomial Representations of GL_n," Lecture Notes in Mathematics, Vol. 830, Springer-Verlag, Berlin/Heidelberg/New York, 1980.

10. J. A. GREEN, The characters of the finite general linear groups, *Trans. Amer. Math. Soc.* **80** (1955), 402–447.
11. N. IWAHORI, On the structure of the Hecke ring of a Chevalley group over a finite field, *J. Fac. Sci. Tokyo Univ.* **10**, No. 2 (1964), 215–236.
12. G. D. JAMES, Unipotent representation of the finite general linear groups, *J. Algebra* **74** (1982), 443–465.
13. G. D. JAMES, "Representations of general linear groups", London Math. Soc. Lecture Notes 94 (1984), Cambridge.
14. G. D. JAMES, "The irreducible representations of the finite general linear groups", *Proc. London Math. Soc. (3)* **52** (1986), 236–268.
15. P. LANDROCK, Finite group algebras and their modules, *London Math. Soc. Lecture Notes Ser.* **84** (1983).
16. B. SRINIVASAN, "Representations of Finite Chevalley Groups," Lecture Notes in Mathematics, Vol. 764, Springer-Verlag, Berlin/Heidelberg/New York, 1979.

Printed in Belgium